

University of Alberta

The Tristram Shandy Paradox

by

Nikki Michelle Albert



A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment
of the requirements for the degree of Master of Arts

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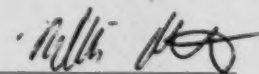
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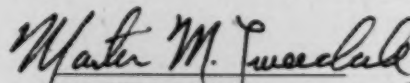
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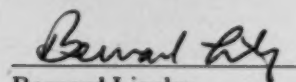
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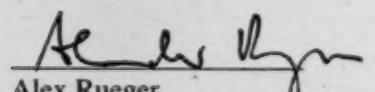
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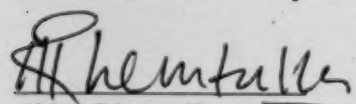
Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled The Tristram Shandy Paradox submitted by Nikki Michelle Albert in partial fulfillment of the requirements for the degree of Master of Arts.


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To my Dad, for his philosophical mind, his insight into the paradoxes of time and for the many enjoyable conversations we had on this topic. And for my mom, thank you for the endless support and encouragement.

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Abstract

This work is an analysis of William Lane Craig's version of the Tristram Shandy Paradox which purports to prove an actual infinite temporal regress is impossible. The Tristram Shandy Paradox sets up an illustration where we are to imagine Tristram has been writing his autobiography for all eternity. Furthermore, the paradox states Tristram writes so slowly it takes him a year to record the events of one day. From this paradox William Craig claims there arises two contradictory results: 1) Cantor's principle of correspondence leads to Tristram Shandy finishing his autobiography by the present. 2) At the same time he has fallen infinitely far behind. I argue against both these contentions. I aim to demonstrate that the paradox itself presupposes a beginning and not that an actual infinite of past time is impossible.

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The ordinal number of an enumerated infinite- (Omega)	ω
The Cardinal number of a denumerable infinite- (aleph-null)	$\aleph_0$

Introduction:

This thesis is the result of my tentative research into the eternity of time and motion. I came to the realization that the focus of the material on this issue revolved around the existence of an actual infinity. If it can be demonstrated that the history of temporal events would construct an actual infinity and that entails various absurdities then past time cannot be infinite. In the sense that a potential infinite can be defined as capable of indefinite increase but remains at any time finite, an infinite past could not be thus defined. If the whole of the past is finite but increasing indefinitely, then clearly it has a beginning with no end. Therefore if the past has no first member then it would construct an actual temporal infinity.

The terms 'potential' and 'actual' can lead to some confusion. If we use the familiar sense of potential to refer to objects not actualized or not yet in existence then we might be led to believe the past would construct a potential infinite whose members are no longer present. This potential/actual distinction is found in Aristotle. One of the ways Aristotle describes a potential infinite is "being infinite in the sense that the parts exist successively, in contrast to which the actual is that whose parts exist simultaneously." (Wolfson, p. 412). Aristotle states that a potential infinite is one which "One thing is always being taken after another, and each thing that is taken is always finite, but always different." (Wolfson, p. 412). Aristotle further states that time and movements are infinite "in the sense that each part that is taken does not remain." (Wolfson, p. 412). This allows Aristotle to say that time is infinite because it is instantiated successively and the parts do not remain, whereas an actual infinite is given all at once and implies an infinite amount of objects in existence in

the present moment. As the parts of time and past events are no longer in existence they do not form an impossible collection as an actual infinite number of any object in the present clearly would. This distinction is important because an actual infinite number of objects instantiated all at once in the present is definitely different than an infinite number of past events and moments. However, that is not the distinction we want to focus on for this paper.

A potential infinite, in the sense we are interested in, is a series that can be increased indefinitely by repeating the operations involved in its production but which remains finite at any time the series is halted. The series of integers starting from zero and progressing successively is such a series, for no matter how far we progress the series remains finite. A potential infinite then is always a finite collection that is capable of indefinite increase. An actual infinite is greater than any assignable number and thus greater than any finite quantity. The set of all integers would be such a collection. An actual infinite can have a first term (1, 2, 3,...) or a last term (...-3, -2, -1), or it may have neither a beginning nor an end (...-3, -2, -1, 0, 1, 2, 3,...). Past events are definite members that can be collected into a set and if that set has no first member then it would be an actual infinite. The past temporal series having no first term and having a last term would construct an actual infinite. Starting from the present moment and regressing into the past would create a finite interval only, and no matter how far we regressed we would never reach a first term. However, the past is an actual infinite because an entire infinite series of past events have been completed up to the present. From the present progressing into the future, however, would only construct a potential infinite. For no matter how far we progress

into the future the interval between the present and that future time will always be finite. Only a finite amount of events or days would be passed through and that finite interval would never progress into an infinite. Thus the issue has been whether an actual temporal infinite is indeed possible in reality. The claim is that an actual temporal infinity is impossible because it leads to various absurdities and paradoxes.

The possibility of an actual temporal infinite has long been a matter for dispute between creationists and those who supported the eternity of the world. In the present some may claim that it is a forgone conclusion that all the components of the universe had a beginning, an assumption that becomes strengthened by the argument that an actual temporal infinity is impossible. This leaves those of us that may want to acknowledge the possibility of infinite time and motion in the awkward position of trying to demonstrate that an actual infinite is not absurd. When it comes to an infinite collection it certainly seems that finding paradoxes is easy but solving them conclusively is the far more difficult task. I intend to carefully analyze just one such paradox and decide if it does entail that an actual infinite past is absurd.

In particular, I intend to focus on a contemporary paradox initially raised by Bertrand Russell called the Tristram Shandy paradox. According to Russell's story Tristram Shandy is a character that takes an entire year to record the events of a single day of his autobiography.¹ William Lane Craig revives the paradox and brings up further absurdities when the paradox describes an actually infinite past instead of a

¹ References to Russell's account of the Tristram Shandy Paradox taken from Bertrand Russell's *The Principles of Mathematics*. (W.W. Norton & Company, Inc.; New York, 1943) p. 355-364.

potentially infinite future.² Chapter One will be a brief account of the historical context from which this argumentation arises.³ Chapter Two will include the modern discussion of this particular paradox with attention on Russell's version of it. In this section we will explore Cantor's Principle of Correspondence and the axiom that the whole is greater than the part, both of which were Russell's concern. Chapter Three will begin with William Craig's version of the Tristram Shandy Paradox, which arises from the Cantorian Principle. This chapter will also include various responses to the paradox. Chapter Four will clear up the major concerns of the previous chapter. Finally, Chapter Five will be the ramifications and conclusions resulting from the discussion.

² William Lane Craig raises the Tristram Shandy Paradox in: *The Kalam Cosmological Argument* (Macmillan Press Ltd.; London, 1979); *Theism, Atheism, and Big Bang Cosmology*. Written with Quentin Smith (Clarendon Press; Oxford, 1993); "Time and Infinity" (International Philosophical Quarterly; 31, 1991, p. 225-231) and "Reply to Smith: On the Finitude of the Past" (International Philosophical Quarterly; 33, 1993, p. 225-231).

³ Chapter One takes information from Herbert Davidson *Proofs for Eternity, Creation and the Existence of God in Medieval Islamic and Jewish Philosophy*. (Oxford University Press; Oxford, 1987), William Lane Craig *The Kalam Cosmological Argument* (1979), and Harry Wolfson *The Philosophy of the Kalam* (Harvard University Press; Cambridge, Massachusetts and London, England, 1976).

Chapter One: Kalam Cosmology

During the Medieval period the argument for the eternity of the world focused on demonstrating that an actual temporal infinity was conceivable. Creationists offered three basic arguments why the concept of an infinite past was absurd. Their arguments were based on three facts: 1) the infinite cannot be traversed, 2) the infinite cannot be added to and 3) infinite series' cannot be put into ratio with each other. The primary source of this argumentation is found in Kalam Cosmology in an effort to demonstrate that since an actual infinity leads to absurdities the world must be finite and created. The Muslim school of thought called the Kalam was interested in cosmology and "developed various forms of the argument from temporal regress." (Craig, *Kalam Cosmological Argument*, 1979). The Kalam school of thought developed in the Middle Eastern nations in the 9th to the 12th century and in Muslim Spain from the 10th to 12th century.⁴ The Mutakallimun⁵ adhered to the notion of creation *ex nihilo*, although they argued other orthodox doctrines, and thus were against the Aristotelian notion of the eternity of the world. These concerns, far from being decided definitively in the medieval period, are the same ones that we have today.

(1.1) The Infinite Cannot be Traversed

One feature of the infinite is that it is boundless and the traversal of it seems impossible. The series of whole numbers is a series that we know is infinite and it is

⁴ William Lane Craig *The Kalam Cosmological Argument*, 1979.

precisely this that makes us distinctly aware that we would be unable to count all the numbers in such a series and thus we are unable to reach the end. When a series is instantiated in time, the problem becomes that we find we would be unable to ever reach the present. Yet clearly we are in the present, which may imply that the past is finite.

We may be immediately tempted, when faced with the problem of traversing the infinite, to reply that due to the infinite divisibility of time and space we traverse the infinite with each step we take and with each passing minute. However, just because time is infinitely divisible does not mean it is actually ever infinitely divided. Time can be potentially divided infinitely, but it is never actually infinitely divided. Time is "infinitely divisible, in the sense that division can proceed indefinitely, but time is never actually infinitely divided, neither does one arrive at an instantaneous point." (William Craig, *Atheism, Theism, and Big Bang Cosmology*, p. 29). One response, then, to the problem of infinite divisibility is to claim that time "as a whole duration is thus logically prior to the (potentially infinite) divisions we make of it." (Craig, *ibid*, p. 29). In addition, "instants are not temporal intervals, but merely the boundary points of intervals, which are always non-zero in duration." (Craig, *ibid*, p. 29). Another response is to claim that time is composed of time atoms of a finite duration, and thus time is not infinitely divisible as time is composed of indivisible time atoms. Regardless of how we might want to address the issue of time's divisibility it does not appear off hand to solve the problem of traversing the infinite.

⁵ "A scholastic theologian or a practitioner of Kalam was called a Mutakallim." (William Lane Craig, *The Kalam Cosmological Argument*, 1979).

For passing from one minute to another does not constitute the traversal of an actual infinite, instead it is the traversal of an interval that can be indefinitely divided.

The Islamic philosopher Ishaq al-Kindi (800-873) brings up an argument based on the traversal of the infinite. It runs thus; "[If past time were infinite], then before every segment of time there would be another segment, *ad infinitum*. But in that case no given time could ever be reached. For [the duration] from the infinite past to the given time would be equal to the duration ascending back from the given time... to infinity... If something is infinite, its interval cannot be traversed."

(Davidson, p. 107). This is because any given moment cannot arrive unless the one previous has arrived, but in the case of an infinite number of moments an infinite amount would have to be traversed to reach the present. Al-Kindi states that the duration from the present moment back into infinity is the same duration from past infinity to the present moment. His reasoning is that if we know the duration from past infinity to now, then likewise we know the duration from now back to infinity. This would make the infinite finite, as it would have known limits. William Craig notes, while explaining Al-Kindi's reasoning, that to select any given moment in time we must know what time it is and when we know what time it is "then we know how long it has been from that moment back to eternity. So time is not infinite at all, since we know how much has transpired. The assumption, of course, is that to select any given moment we must know that moment in relation to the past time before it instead of in relation to the present moment in time," (William Craig, *Kalam Cosmological Argument*, 1979). Al-Kindi appears to believe as if there is an infinity point in the past, and therefore if we know how far the duration from the present to that point is

then it is indeed not infinite. And from this infinieth point we would have traversed an infinite interval to reach the present, which seems an impossible feat.⁶ Firstly we would have to point out that we could not find an infinieth point, even if one could exist, and therefore we would not be able to know the interval from this point to the present. Al-Kindi believes that if we could find such a point this would imply the interval is finite. This reasoning implies that Al-Kindi believes that if the interval has both a first member and a last member that the segment will be finite and not infinite. We must acknowledge that there may be infinitely distant events in the temporal series, and from that infinitely distant event to the present would construct an infinite interval, not necessarily a finite one. Therefore if there are infinitely distant events we still have the problem of traversing this distance to the present, regardless of the fact that we cannot locate the event or know the duration from it to the present. If there are not infinitely distant events, and no infinieth point to be found then from any given moment we can locate in the past to the present would be finite and we would not necessarily have to traverse any infinite distances to reach the present.

The traversal of the infinite is a problem that plagues many philosophers. Each one uses essentially the same argument with different examples or emphasis. The Jewish philosopher Saadia⁷ (882-942) for instance states that it is impossible to mentally traverse the infinite backwards, and likewise it is impossible for existence to

⁶ Al-Kindi's traversal argument found in W.L. Craig's *The Kalam Cosmological Argument*. (p.22)

⁷ Saadia's argument found in William Craig's *Kalam Cosmological Argument* (p. 37-41), and Herbert Davidson's *Proofs for Eternity, Creation and the Existence of God in Medieval Islamic and Jewish Philosophy* (p. 95-105).

traverse it actually forwards. Philosophers such as Nazzam⁸ (775-845) noted that that which has no beginning could have no end, and if the present is the end of the temporal series then the past must be finite. One standard response to the problem of traversing the infinite is to state that the infinite cannot be traversed in a finite amount of time, but can be traversed in an infinite amount of time. Aquinas (1225-1274) brings up a response that suggests that an entire infinite interval does not have to be traversed, only a finite portion. For "Though the infinite is not [traversed] when simultaneous and actual, it can be [traversed] when in succession [and in potentiality], since any part of such infinities taken is finite." (Wolfson, p. 461). According to Aristotle "an infinite cannot be traversed refers only to a continuous infinite extension which has neither beginning nor end, but in the case of an infinite by succession... means that 'one thing is always taken after another, and each thing that is taken is always finite,' there are parts which are traversed." (Wolfson, p. 461-2). Between any two given days the interval is finite and therefore "Whatever bygone day we choose from it to the present day there is a finite number of days which can be traversed." (Wolfson, p. 462). Therefore, all that is traversed is a finite segment and then the segment recedes into the past as well. Basically the whole infinite is not traversed because the infinite is successive in nature and is traversed term by term. However, if within the series there are infinitely distant events then there would remain the problem of traversing an infinite interval.

⁸ References to Nazzam found in Herbert Davidson's *Proofs for Eternity, Creation and the Existence of*

(1.2) The Infinite Cannot Be Added To

The idea that you cannot add to an infinite collection seems to arise from the idea that the infinite is the greatest number instead of greater than any assignable number. The intuition then is that if you have a series that is infinite and add one more member you get something that is greater than the infinite. However, if we take the infinite to be greater than any assignable number, when we add to the infinite it remains infinite. The series of whole numbers would construct an actual infinite because it has no last term and it contains all the whole numbers. As it is actually infinite and contains all the whole numbers it seems there are no whole numbers left to add. The reasoning behind this is that if you add to any magnitude, including an infinite magnitude, it should then be increased in size. Therefore if by adding to the infinite it fails to be increased this is seen to be absurd. However now we would say we can add to this series by simply starting over (1, 2, 3, ... 1, 2). By adding to the series we have not created something greater than the infinite because the infinite is greater than any assignable number, and therefore we can continue to add to this series without creating something greater than the infinite.

Al-Kindi demonstrates the impossibility of adding to an infinite based on some simple principles. If two bodies are added together then the body that is created will be greater. If two finite bodies are combined the result will be finite. Now if you have an infinite magnitude and remove a finite amount the result will either be finite or infinite. It cannot be finite, "for when the subtracted portion is restored, the total, as the sum of two finite magnitudes, would also be finite," (Davidson, p. 108). It

cannot be infinite and yet "fail to be increased when the subtracted portion is restored; for every magnitude undoubtedly becomes greater when another magnitude is added to it." (Davidson, p. 108). Nor "could the remainder be infinite and yet... be increased through the restoration of the finite portion, for 'there cannot exist two infinite magnitudes of the same type one of which is greater than the other.'" (Davidson, p. 108). Al-Kindi further states that the smaller infinite would be equal to part of the whole. Al-Kindi believes that if the smaller infinite is contained within the whole infinite then the part would have limits and thereby cannot be considered infinite. The smaller infinite also cannot be equal to the combined infinite, because that would result in the whole being equal to one of its parts. What is important to note here is that the principle that the whole is greater than the part does not seem to be valid in regards to infinite quantities. Al-Kindi wants to hold onto this common sense principle and therefore thinks it impossible that the 'parts' of an infinite could be equal to the 'whole'.

Saadia also states that with each passing day the infinite past would be added to and as the infinite cannot be so increased past time must be finite. Saadia phrases this as each day increasing the past and diminishing the future. And "whatever admits increase and diminution is of finite power." (Davidson, p. 98). Saadia mentions the diminution of the future because he wants to point out that the future is finite as well. This is brought up by Shahrastani, as he states "if we took off a year from the past and added it to the future, then the past would be decreased and the future would be increased, even though they are both equal in infinity, whence it would result that the more would be equal to the less." (Wolfson, p. 424). Saadia's

point that what is capable of increase and diminution is finite applies to a potential infinite, since no matter how much one increases the series it remains finite.

(1.3) Infinite Series' Cannot Be Put Into Ratio with Each Other

The argument that infinite series' cannot be put into ratio with each other often occurs along with the argument that the infinite cannot be added to. The argument is usually illustrated by the revolutions of various celestial spheres found in Medieval Astronomy. If the universe is eternal then each celestial sphere has completed an infinite amount of revolutions. And yet, for example, the Sun will have thirty times more revolutions than that of Saturn every year, and that implies that the series of Sun revolutions would be thirty times greater than the series of Saturn revolutions. Not only is each infinite series added to by additional completed revolutions but one infinite series is thirty times greater than another infinite series. The puzzle that arises from this is that, that while both series' being compared are equally infinite, one is at the same time greater than the other.

Ghazali (1058-1111) notes that if you wish to side step this argument by claiming the sphere's revolutions cannot be numbered, there is still a clear ratio between them that must be acknowledged. In other words they do stand in a numerical relation, even if you cannot compare 'part' to 'part'. Ghazali states, "just as the number of the revolutions of Saturn is infinite so also the number of the revolutions of the Sun is infinite, despite the fact that the former is one thirtieth of the latter." (Wolfson, p. 422). Yet there remains the claim that they are both infinite and cannot be compared, because neither series is complete. The series is complete in the sense that the present would be the end of the series. It is not complete, however, in

the sense that is open-ended or infinite in the past. Since we cannot establish a beginning we are unable to begin a comparison of parts. This is precisely the point raised by Averroes. Averroes claims that given that the infinite revolutions of any sphere are not a completed whole with definable limits, there can be no ratio established. Averroes being aware that Aristotle stated "the infinite is that of which, however much one has taken, there is always more to take, whereas that of which there is nothing more to take is complete and whole." (Wolfson, p. 431) and the infinite unlike a completed whole lacks at least one end or limit. If the series were finite then indeed we could make the claim that one series is greater than the other. We would simply compare each part of both series from the beginning to the end. There cannot be a comparison in sizes of infinite series if neither is a completed whole. Two infinite series cannot be compared part-to-part, since it makes no sense to speak of parts when there is no completed whole. Alfarabi (870-950) further explains that past motions continue to exist in the imagination "since they do not exist *together* in actuality, they cannot properly be enumerated and therefore are not affected by the absurdity of an actual infinite number." (Davidson, p. 128). As Ghazali notes there is a ratio present regardless of whether we can compare the series' part-to-part. This is because we are aware that in the same span of time one sphere makes thirty more revolutions than the other, and this correlation enables us to state that one series is thirty times greater without comparing each part of the series from a determined beginning. However, if we want to compare the sizes of infinite series we cannot do so based on finite notions of wholes and parts, nor can we do so by counting each member individually.

(1.4) Summary

As we can see from the preceding discussion there are many problems that complicate the idea of an infinite collection. The problem of traversing the infinite seems to imply that we would have to traverse an infinite amount of moments to reach the present moment. If a series has infinitely distant events then we are confronted with this notion of traversing an infinite to reach the present. But if there are no infinitely distant events then we find that any point in the past to the present is finite. The fact that an infinite collection of moments has receded into the past seems to suggest that all that is traversed is finite terms successively instead of an infinite progression of moments; each moment requiring the one previous to it, and so on *ad infinitum*. There seems to be no easy answer to the notion of traversing the infinite as it arises from our awareness of the infinite being endless. As we are aware of our inability to count to infinity, the inability of ever completing an infinite series, likewise we find it incomprehensible that a series with no beginning can have an end.

Secondly if the infinite is described as the greatest number conceivable then it seems odd that we could add to that creating something greater. Actually the infinite should be described as greater than any assignable number and therefore adding to an infinite does not create something greater than an infinite.

Medieval ideas seem to be confused by the axiom that the whole is greater than the part, which clearly holds of finite collections. If the world is eternal then the sphere of the Sun would have made an infinite amount of revolutions. However, the

Sun would have completed an infinite amount of revolutions a hundred year ago as well. That smaller section is infinite as is the whole series. Reason tells us that the whole series should be greater than the smaller section. However the subset in this case is not a completed whole, and neither is the entire set, thus both being infinite they can be considered equal in size. Medieval thinkers considered the notion that in an infinite series the whole is equal to its parts is absurd, and paradoxes thus arise from accepting it. We can disregard the principle simply because infinite collections, unlike finite collections, are not intervals that construct a finite whole with two ends, and thus the axiom does not apply to them. If we take the sphere of the Sun and the sphere of Saturn both sets of revolutions separately create their own infinite series and operate under a ratio of one to thirty. It can be replied that because neither series has a first term we cannot establish a ratio. However, the medieval thinkers believed that the Sun completed thirty revolutions to Saturn's one completed revolution a year. Therefore, it was reasonable to conclude that the series of Sun revolutions is thirty times that of the series of Saturn revolutions. At the same time, if both series are infinite then they are also the same size. The principle that the whole is greater than the part does not seem to be valid for infinite collections. A section of an infinite series can be equal to the whole series when the series is infinite. Regardless of the fact that the series of Sun revolutions appears to be thirty times greater than the series of Saturn revolutions both of them construct a denumerable (or countable) infinite and therefore are equivalent. We cannot claim that one series is both greater than another and equal to it at the same time, for as the medieval philosophers note this is what makes the situation absurd.

Chapter Two: Cantor and Russell

Russell realized that the axiom 'the whole is greater than the part' creates problems when applied to the infinite. He also knew that we could not apply the axiom and the Cantorian principle of correspondence at the same time. For the principle of Correspondence implies the whole and the part are similar when applied to infinite sets. It is using the Cantorian Principle of Correspondence that Russell believes he can eradicate the paradoxes of the infinite.

(2.1) Cantor⁹

Cantor establishes the principle of correspondence in order to compare the cardinal number of infinite sets. We can reasonably state whether one set is larger than another by correlating the members of each series. It would be impossible to count all the members in each infinite set to establish which one has more members but by creating a correspondence between members we can find out if one set has members that cannot be paired off with a member from the other set. If there are members that must be left over in either set, no matter what sort of correspondence between terms is proposed, then one set is indeed larger and therefore has a different cardinal number. In this correlation sense then one infinite set can be 'bigger' than another. A denumerable infinite is an infinite series that can be correlated to the series of integers; therefore all denumerable infinite collections are equivalent. Two sets are equivalent, as Russell mentions, if; "u is similar to a proper part of v, and v to

a proper part of u (a case which can only arise when u and v are infinite), then u is similar to v ; hence 'u is greater than v' is inconsistent with 'v is greater than u.'"
 (Russell, p. 122). Russell uses the term 'similar' to determine that u has the same cardinal number as a proper part of v and vice versa, thus u is equivalent to v . The cardinal number of the set is used to show how many members the set contains without referring to any particular order the members are in. One set has the same cardinal number as another set just in case all the members of one can be put into a one-to-one correspondence with all the members of the other (A set which has the same cardinal number as the set of integers is denumerably infinite, i.e. its cardinal number is $\aleph_0$). Cantor proves that occasionally infinite sets can have different sizes.¹⁰
 The set of real numbers (between 1 and 0) and the set of natural numbers do have a different size because some real numbers do not pair up with a natural number. Therefore we can say that the infinite set of real numbers is greater than the infinite set of natural numbers.¹¹ Some philosophers argued that the set of real numbers cannot be paired up with the set of natural numbers because we have simply not discovered a method to pair off these terms.¹² For our purposes if we have two denumerable sets with the cardinal number $\aleph_0$ (aleph-null) they are equivalent.

It is through the correlation between members of infinite sets that we are led to the denial of the axiom that whole is greater than the part. We find, for example, the

⁹ A. W. Moore p. 110-128, Russell's *Principles of Mathematics*, and Fraenkel's *Abstract Set Theory*.

¹⁰ Were "it not for the existence of different infinite sizes, transfinite mathematics would be the merest shadow of the subject it in fact is." (Moore, p. 119).

¹¹ Moore, p. 118-120.

¹² Henri Poincare "Cantor's proof could just as well be taken to establish merely that we could not devise a way of pairing off the natural numbers with real numbers," (Moore, p. 121-22) Others thought that the real numbers did not construct a genuine set at all, "presumably because the real numbers were somehow too unwieldy to be grouped together into one determinate totality." (Moore, p. 122)

set of whole numbers is equivalent to the sub-set of even whole numbers. It appears as though the set of even whole numbers would be smaller, because intuitively we might think it is half the size of all the whole numbers. However, when we correlate both sets there are no members that cannot be matched up and thus the sets are equivalent. For every term n in the set of whole numbers there is a term $2n$ found in the sub-set of even whole numbers. Given that both sets have no last member this is really not a surprising result, for they are clearly both denumerable infinite collections. It is only counter-intuitive if you think of them as finished in the finite sense of having both a first term and a last term, and conclude that therefore the set of even whole numbers would be half that of the set of whole numbers. As an infinite collection does not have a finite number of members and does not commonly have two definite ends it is not surprising that the axiom of the whole is greater than the part does not apply. So in infinite sets the entire set is equivalent to its infinite subset. Or as Russell states the whole is similar (equal) to the part. We do have a whole or completed series with the series of whole numbers (1, 2, 3 ...) which has as its ordinal ω , because the series has all the whole numbers as its members. As Moore states we can still claim that the subset is smaller in the subset sense if not in the correlation sense.¹³ The axiom that the whole is greater than the part is only valid for finite collections, because only finite collections create a whole suitable for that axiom. A finite whole then provides a finite collection with both a last term and a first term. The axiom the whole is greater than the part is valid for finite whole collections. An infinite whole requires that a series lacks a first member, a last member or both, but

¹³ Moore, p. 111.

contains all the members that belong to that series. The contrary axiom that the whole is equal to its parts is valid for infinite sets and the set's infinite subset or proper parts.

William Lane Craig wonders if the Cantorian principle of correspondence also only applies to finite collections. He states that clearly the principle of correspondence and the axiom of the whole is greater than the part cannot both be true because "one asserts that the whole is greater than a part, while the other maintains that the whole is not greater than a part." (Craig, *Kalam Cosmology*, p. 95). Using Cantor's principle of correspondence we can establish that the set of all natural numbers is equivalent to the set of even natural numbers. Craig notes that an interesting feature of this correspondence is that "the number in the second series is always the double of the number in the first series." (Craig, *ibid*, p. 73). I think Craig is alluding to the extensiveness of the set of whole numbers compared to its subset of even numbers. Having, say, counted to ten, I have numbered 10 terms that belong to the set of whole numbers but only 5 terms that belong to its subset of even whole numbers. The set of whole numbers acquires more terms (twice as much) than the set of even whole numbers as we count. With the principle of correspondence Cantor can assert that ω and $\omega + \omega$ both have the cardinal $\aleph_0$. Despite the fact that the set of whole numbers increases its members at twice the rate of the even whole numbers the set can be correlated term for term with its subset. William Craig has difficulty comprehending how a set that contains all the even whole numbers and all the odd whole numbers can be equivalent to a set that contains only the even whole numbers. Furthermore he claims it is the fact that $\omega + \omega$ has the same cardinal number as ω that

creates problems with infinite collections found in the real world. If an actual infinite were found in reality the set $(1, 2, 3, \dots)$ would be equivalent to a set with $(1, 2, 3, \dots, 1, 2, 3, \dots)$. Craig then concludes that it is more reasonable to suggest that both principles are valid in reality and that actual infinite collections are impossible in reality.

Therefore the principle of Correspondence and the principle the whole is greater than the part are both valid for finite collections found in reality, and infinite collections are reserved for the mathematical realm. Russell states that there "is no evidence for the axiom except supposed self-evidence, and its admission (to infinite collections) leads to perfectly precise contradictions." (Russell, p. 360). Russell continues, "the similarity of whole and part could be *proved* to be impossible for every *finite* whole, it became not unplausible to suppose that for infinite wholes, where the impossibility could not be proved, there was in fact no such impossibility." (Russell, p. 360).

Therefore although the axiom the whole is greater than the part is invalid for infinite collections we can still assert that the whole is similar to the part. Since the principle of correspondence entails that the whole and part are similar it is valid for infinite collections whether they be in the mathematical realm or found in reality. Russell was attempting to discover whether the axiom that the whole is greater than the part was valid or the principle of correspondence was valid for infinite collections. Russell concluded that the axiom the whole is greater than the part created paradoxes and the principle of correspondence did not. Craig's motivation then is to demonstrate that the Cantorian principle also leads to absurdities when applied to an actual infinity.

(2.2) Bertrand Russell

In *The Principles of Mathematics* Russell brings up the original Tristram Shandy illustration. To figure out if the Cantorian principle creates any contradictions as the axiom 'the whole is greater than the part' does when applied to the infinite, Russell constructs the Tristram Shandy illustration. Tristram Shandy is a character that is writing his own autobiography. He writes so slowly that it takes him an entire year to record the events of a single day. It becomes apparent that Tristram will very quickly fall behind in his progress. Therefore he will never be able to complete his work. Russell then claims; "Now I maintain that, if he had lived forever, and not wearied of his task, then, even if his life continued as eventfully as it began, no part of his autobiography would have remained unwritten." (Russell, p. 358). In Russell's account the set of days recorded and the set of years of writing each have a first term but with no end. As each series has a first term it is easy to pair years to days in a correspondence. Thus every day will have a year of writing matched to it, and every day will be written about. There being a one-to-one correspondence between terms means that the sub-set of days recorded is equal to the set of years of writing. Russell suggests that this means the whole is similar to the part. Russell concludes that the Cantorian principle is valid for infinite sets, since unlike the axiom of the whole is greater than the part, no contradictions arise from its use.

Figure (2.1) Russell's Version of the Tristram Shandy Paradox

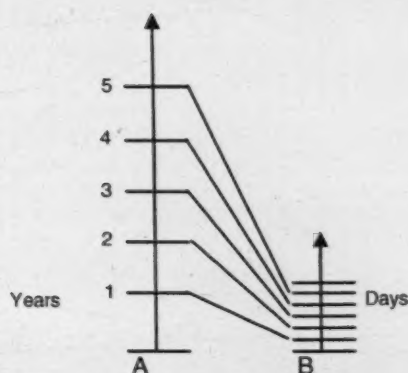


Fig. 2.1. Fig 2.1 illustrates how correspondence can be established between the set of year of writing (A) and the set of days recorded (B). In Russell's version of the Tristram Shandy Paradox a correspondence between members is easily established as both sets have a first member. As we can see, even though each series diverges more and more the terms can be matched up in a one-to-one correspondence that enables us to know they are equivalent sets. In addition, if we halt set A at year 5, five days of set B will have been recorded but as they are not temporally lined up Tristram will clearly not be finished writing his autobiography. However, it is true that no day will remain unrecorded for there will always be a further year in the progression to write another day.

There is no problem with the statement that each day recorded can be correlated to a different year of writing. We can see this from figure 2.1 above that with the correspondence between years and days both sets have the same number of members. Both of the sets are increasing their members at a ratio of 1:1 (one day recorded for every one year of writing) and therefore will always be equivalent. This does not entail that Tristram will be finished writing or that he will be caught up at any moment, it only demonstrates that both infinite sets are the same size and are increasing at a determined rate. Russell's version of the Tristram Shandy paradox is illustrated as progressing from the present into the future and therefore as the series has a first term but no last term it constructs a potential infinite. Because the series is potentially infinite there will always be a year further into the future to record another day. As time passes, the further we progress into the future the more days become

recorded. And since we can potentially progress into the future indefinitely there is no past day in the series that will *remain* unrecorded. Russell concludes; "the Tristram Shandy proves that two variables which start from a common term, and proceed in the same direction, but diverge more and more, may yet determine the same limiting class." The set of years of writing is more temporally extensive than the set of days recorded; even though the sets diverge more and more they are equivalent sets that are indefinitely increasing at the respective rates of one day per year. As Russell claims, the correspondence between terms does not entail any absurdities. Nor did Russell intend his illustration to create absurdities; in fact he felt that it demonstrated that the principle of correspondence is valid for infinite sets.

It is also true that given any point in the series the correspondence holds but we can clearly see that Tristram is behind and not caught up in his writing. If we look to figure 2.1 again we can clearly see that temporally set A is more extended than set B. If we halt set A at the end of year 6, we can see that set B will not be temporally lined up to set A and therefore there will be days that are not yet recorded. There will be days that Tristram has lived through but are not yet a member of the subset of recorded days. These unrecorded days will be later correlated to future years. As set A increases by one member so will set B, and thus a previously unrecorded day will be added to set B. Therefore, the correspondence, that enables us to know the sets are the same size, does not lead to the conclusion that both sets are matched up in duration. So from within the series, not every day is recorded. Looking at the set from within a finite portion of the series enables us to see how it is instantiated in time and this tells us that at any point Tristram will not be finished writing. We realize he will

never finish writing. Therefore correspondence does not entail that Tristram is finished writing at any time.

Russell's conclusion that the whole is similar to the part may be misleading if we think that 'whole' is necessarily a finite term and entails a series with two definite ends. However, Russell makes a distinction in the definition of a 'whole'. He distinguishes "wholes which are defined extensionally, i.e. by enumerating their terms, from such as are defined intensionally, i.e. as the class of terms having some given relation to some given terms, or, more simply, as a class of terms." (Russell, p. 349). The whole that is defined extensionally is necessarily finite. A "finite whole may be taken collectively, as such and such individuals A, B, C, D, E say. A part of this whole may be obtained by enumerating some, but not all, of the terms composing the whole; and in this way a single individual is part of the whole." (Russell, p. 360). Finite wholes and parts may also be defined by intension. Thus "we know without enumeration that Englishmen are part of Europeans; for whoever is an Englishman is a European, but not *vice versa*. A whole defined intensionally does not require enumeration and can be an infinite collection that refers to "a whole whose parts are the terms of a class." (Russell, p. 349). A whole taken intensionally then does not refer to a collection with definite limits but to a class which definite items either belong to or not. It is as such defined "without any reference to any specified individual" (Russell, p. 349) and without enumeration and thus a whole in this sense can be infinite. If "*a* be a class-concept, an individual of *a* is a term having to *a* that specific relation which we call a class-relation." (Russell, p. 360). If "now *b* be another class such that, for all the values of *x*, '*x* is an *a*' implies '*x* is a *b*' then the

extension of a (i.e. the variable of x) is said to be *part* of the extension of b ." (Russell, p. 360) Here there is no enumeration of individuals needed. To say that a and b are similar is to say "that there exists some one-one relation R fulfilling the following conditions: if x be an a , there is a term y of the class b such that xRy ; if y be a b , there is a term x' of the class a such that $x'Ry$ " (Russell, p. 361). Russell gives the example of prime numbers in which primes are the proper part of integers, but this cannot be established by enumeration. It "is deduced from 'if x be a prime, x is a number', and 'if x be a number, it does not follow that x is a prime.'" (Russell, p. 361). That the class of primes "should be similar to the class of numbers only seems impossible because we imagine whole and part defined by enumeration." (Russell, p. 361). Regardless, we are able to say that the infinite subset correlates equally to the infinite set.

Seeing the infinite as whole or complete is not a universally accepted notion. Some people see the infinite as progressing toward a limit that it never reaches, for every member the set receives there is always more to take. As such they acknowledge the validity only of the potential infinite. Any series or set is seen as able to continually acquire more members without becoming complete, and therefore increases indefinitely. According to Cantor the potential infinite is considered the improper infinite. An actual infinite is considered the true infinite, we "meet the true infinite when we regard the totality of numbers 1,2,3,4, ... itself as a completed unity, or when we regard the points of an interval as a totality of things which exists all at once." (Craig, Atheism, Theism and Big Bang Cosmology, p. 7). Rather "than attempt to synthesize the infinite mentally by counting, Cantor stood outside the

infinite series of natural numbers and grasped them conceptually as a totality." (Craig, *Atheism, Theism and Big Bang Cosmology*, p. 11). When we conceive of the infinite from beyond the series itself we can understand an infinite series as complete or whole. The series of whole numbers is considered a complete infinite series, for the series contains every one of the whole numbers and there are no whole numbers remaining. Even though we can not count every member of that series we know that every whole number is contained within it. Therefore we can understand the infinite as a whole, or as a series that is always incomplete and indefinitely taking on more members.

Both viewing an infinite collection as a whole and viewing it as incomplete have concerns associated with them. For instance if the infinite is conceived as a collection indefinitely increasing but remaining at any time a finite collection the problem of traversing an infinite interval is a serious concern. As the infinite is always approaching a limit it never attains, there will always be more members for the collection to collect, and we will never traverse the infinite. This problem can be illustrated by the impossibility of counting to infinity, where no matter how much we count there is always more to go, and we never reach infinity. In addition, if we conceive of an infinite as complete it seems odd that we would be able to add to that collection. As an actually infinite collection already contains within it all its members we have no members left over to add to it. For example the series of whole numbers contains within it all the whole numbers, with none remaining to be added to it. If we do want to add to an actually infinite series of whole numbers we would have to start repeating members, by starting the series of whole numbers again. The infinite set of

whole numbers is complete, but we can always add to the series of whole numbers by starting the series over.

(2.3) P.O. Johnson¹⁴

P.O. Johnson claims that the principle of correspondence does not aid us in clearing up any paradoxes of the infinite and may indeed cloud the issue further. He points out that by applying a correspondence we are not able to state that one series is more extensive than the sub-series. Johnson makes the valid point in saying that since there is always a further member to pair up hides the fact "that the sub-series is using up its terms at a much faster rate." (Johnson, p. 369). Johnson is referring to the series of whole numbers and the sub-series of even whole numbers. Indeed, if the series is stopped at any point, and thus made a finite whole, then we could easily establish that the whole is greater than the part. Johnson states if we halt the series of whole numbers and we count backwards "it is obvious that the even number sequence will reach zero in the fewest terms." (Johnson, p. 369). This fact can be illustrated by the Tristram Shandy paradox in which the set of years of writing is more temporally extensive than the set of days recorded. For every day recorded there are 365 days of writing. This is important to pay attention to how the infinite series is said to be instantiated in reality. For instance, since the set of years of writing is more extensive than the set of days recorded a greater duration of time is traversed in the set of years than the set of days. In the case of Tristram Shandy we are aware that the members of the set of years spent writing and the members of the set of days recorded diverge more and more and will never be matched up in time. If we ignore this fact then we

may be led to the conclusion that because the sets are equivalent the principle of correspondence leads to Tristram Shandy finishing his autobiography, when clearly that would not be the case. The set of days Tristram has lived through (of which the set of days recorded would be a sub-set of) is as extensive as the set of years spent writing and the two sets are also equivalent through the principle of correspondence.

The Cantorian principle merely reflects a way to compare sizes of the infinite. Johnson suggests that Russell makes a stronger claim by stating the principle entails that the whole and the part can be similar, as is expressed by the Tristram Shandy illustration, where the set of years is similar to the set of days. Johnson states;

"There is no paradox in the fact that Tristram could not complete his autobiography at the rate he is going, even if he lived forever; it is only when one comes to consider that, on the other hand, if he were to live forever, he would apparently be able to complete every part of it, that one arrives at the paradox, which is that it seems to be true that he can never finish his autobiography and that no part of it need remain unwritten." (Johnson, p. 371).

Clearly Johnson sees no problem with Tristram Shandy not finishing his work, due to the fact that the set is far more extensive than its subset. However, Johnson seems to be implying that because the whole is equal to its parts that Tristram could be interpreted as finishing 'all' the parts of his autobiography, therefore finishing, and he finds this to be the contradiction. Thus since correspondence entails that the whole and part are equal we are led to the paradoxical conclusion that Tristram is both finished writing and never finished writing. Of course, Johnson fails to notice that while the set of years is equivalent to the set of days recorded, the set of days recorded is only a part of the set of *all* the days Tristram has lived through. The set of all the

¹⁴ P.O. Johnson "Wholes, Parts, and Infinite Collections." (Philosophy; 67, 1992).

days Tristram has lived through also contains all the unrecorded days in his life. Both the set of days Tristram has lived through and the sub-set of days recorded are equivalent to the set of years spent writing. Only the set of days Tristram has lived through is as temporally extensive as the set of years spent writing. Therefore the subset of days recorded is only a *part* of the set of *all* the days, and 'every part' of his autobiography is not recorded. For despite the correspondence found with the set of days recorded and the set of years spent writing there remains another 'part' of the set of all the days Tristram has lived through, that subset consisting of all the unrecorded days.

Johnson points out that once we discuss the infinite, notions of wholes and parts become inadmissible. And he further notes that the series cannot be a whole for it is not completed, thus there is no finished work. The intuition that an infinite is not completed is a common one. Indeed from within a series the infinite is endless. For instance if we begin counting all the whole numbers no matter how far we count there is still more to go. However, we can have a whole infinite series when we are looking at the series from beyond the series itself. If we are not counting the series of whole numbers but instead look at the series itself we are aware that the series is complete in the sense that it contains *all* the whole numbers. In the case of Russell's Tristram Shandy illustration there is not a completed infinite series, rather there is only a potentially infinite series. As such, no matter how far we progress into the future we create only a finite whole with a first term and a last term. Furthermore Johnson states, it is true that if Tristram lived forever we would write about any day, but not

'all' days or 'every' day.¹⁵ Johnson makes the mistake in assuming that the axiom the whole is similar to the part creates the problem of all the days being recorded. At the same time he does not think correspondence leads to this conclusion even though the axiom is entailed by correspondence in infinite sets. He will not be any closer to finishing, but for every year he will write about a day. No matter how many years we progress an equal amount of days will be recorded, although neither series will ever be actually infinite. The one-to-one correspondence and the axiom the whole is greater than the part simply do not lead to the book being finished. The one-to-one correspondence does illustrate that the set of years of writing is equivalent to the set of days recorded.

Johnson also casts doubt on whether a one-to-one correspondence is valid in this situation. The days and years do not really run in a parallel series, in fact the discrepancy between the series increases with each year. At any point, Johnson points out, there are many days that Tristram has lived though that are not recorded and that have no corresponding term.¹⁶ However, I would first point out that that is looking at the series as halted at some given point. And those unrecorded days are not yet part of the 'days recorded' subset, they will not remain unrecorded since there will be another further year of writing beyond. A correspondence can be established, however, by stating that both sets are equivalent we often ignore the fact that one set is more extensive than another, and we can see this is the case with the Tristram Shandy illustration. If we pay attention to the extensiveness of each set we are aware that

¹⁵ The "term 'any' being crucial, and chosen in distinction from the term 'every', for Tristram would certainly not be able to write about every day of his life." (Johnson, p. 371).

¹⁶ P.O. Johnson, p. 372.

regardless of the fact the sets are equivalent the set of days recorded will never temporally line up to the set of years recorded. Thereby we can determine the correspondence does not entail Tristram will be finished writing.

(2.4) William Lane Craig

William Lane Craig implies that Russell's intention with the Tristram Shandy illustration is to demonstrate, with use of the Correspondence principle, that Tristram would be finished his book. Craig states, if Tristram Shandy were "immortal, then the entire book could be completed, since by the method of correspondence each day would correspond to each year, and both are infinite." (Craig, Kalam Cosmology, p. 33). Of course, as we noted, Russell does not state Tristram would be finished writing, only that no day will remain unwritten. As Craig notes the future constructs a potential infinite and the more Tristram writes the further behind he gets; certainly he never approaches a state where he would be finished. Craig is implying that there is a contradiction between the facts that Tristram can be considered finished and at the same time he falls progressively behind. However, although Tristram will never be finished writing any day will eventually be recorded. Correspondence enables us to state the sets are equivalent without the entailment that Tristram Shandy will be finished writing.

(2.5) Conclusion

Russell's version of Tristram Shandy does not seem to create any problems, if taken as a potentially infinite series. He does not state Tristram will be finished, and as we discover he will never be finished. At the same time we can say that no part of his autobiography will remain unwritten as time progresses in this potentially infinite

series. The use of the correspondence principle does not, then, entail Tristram will complete his work. It does entail that the whole is similar to the part in infinite collections.

It is by the fact that the axiom the whole is similar to the part is entailed by the use of correspondence that Johnson is led to the conclusion that Tristram could be finished writing. To prevent this misunderstanding it is important to keep in mind what the axiom means and paying attention to the extensiveness of the sets in question may aid in this endeavor. With Russell's version of Tristram Shandy the set of years of writing is clearly more temporally extensive, and therefore the duration completed by that set will be far greater than the duration completed by the set of days recorded. From this we can determine that even though these two sets start at the same point, anywhere in the series they will not be equal in duration. Secondly, when we state the whole is equal to the part we do not mean 'every part'. If we meant 'every part' then the set of days recorded would have the same extensiveness of the set of years spent writing. This would entail that Tristram is finished writing, and it would also mean for every year of writing he records 365 days. The set of years spent writing is as extensive as the set of all days Tristram has lived through which includes recorded days as well as unrecorded days. The set of days recorded then is equal to the set of years spent writing in the number of members but not in the temporal extensiveness. As there will also be unrecorded days in the set of days lived through that are not members of the subset of recorded days, 'every part' is not recorded. This can be presented with the series of whole numbers as well. The set of whole numbers is equivalent to its subset of even whole numbers. If I were able to count all the even

whole numbers this does not then imply that I have counted all the whole numbers, clearly there remains all the odd numbers. The principle of correspondence and the axiom the whole is similar to the part can not be used to support the claim that Tristram Shandy would be finished writing.

Chapter Three: The Tristram Shandy Paradox and Responses

William Lane Craig sets out to demonstrate that the principle of correspondence and an actual infinity only work in the mathematical realm.¹⁷

According to Craig it is when the principle of correspondence is applied to an actual infinite in reality that various absurdities arise. An actual infinite is a complete infinite series as opposed to the potential infinite which only increases indefinitely remaining at any time a finite collection. His motivation is to demonstrate that an actual infinity cannot exist in the physical world as opposed to the mathematical realm, and therefore when we envision an actual infinity in the real world and apply the principle of correspondence the results are paradoxical.

(3.1) The Tristram Shandy Paradox According To Craig.

Craig alters the Tristram Shandy Paradox to refer to the past to bring out the apparent paradoxes that arise with the use of the principle of correspondence.

Imagine that Tristram Shandy has been writing for an eternity past and that the past is an actual temporal infinite. Craig claims that a one-to-one correspondence can be established between days written about and years of writing. According to Cantor this means that the set of days recorded and the set of years of writing are equivalent sets.

Therefore, for every year of writing there is a day recorded, and every day lived through has a year in which it was recorded. For Craig this entails that given an infinite amount of time Tristram will have finished writing his autobiography. Craig

¹⁷ "What I shall argue is that while the actual infinite may be fruitful and consistent concept in the mathematical realm, it cannot be translated from the mathematical world into the real world, for this

seems to be led to this conclusion based on the fact that if both sets have an equal infinite amount of members this entails all the days in the past will have been recorded. This is based on the claim that "if the principle of correspondence were descriptive of the real world, he would have finished – which is impossible." (Craig, *Kalam Cosmology*, p. 34). Craig states this is clearly absurd since "Tristram Shandy could not yet have written *today's* events down." (Craig, *Theism, Atheism, and Big Bang Cosmology*, p. 33), for today would require at least the next year, if not some year somewhere in the future, to be recorded. In fact we are aware that Tristram would never finish, "for every day of writing generates another year of work." (Craig, *Theism, Atheism and Big Bang Cosmology*, p. 33), and Tristram falls progressively behind. If the principle of correspondence really related to the real world as opposed to the mathematical realm, then Tristram would be finished. Therefore if we are going to accept the existence of an actual temporal infinity of the past and the principle of correspondence we are led, it seems, to Tristram both finishing his autobiography and also falling behind.

If the principle of correspondence entails that Tristram would be finished with his book, then Tristram could have finished at any time in the past. Craig states this is because yesterday and the day before also have an infinite past. If in an infinite amount of time he could have completed his work by the present, then since every past day has an infinite past he could have finished at any point in the past. But if Tristram could be finished at any point in the past then anytime in the past he will not be writing, however, we assumed him to be writing for all eternity. Thus, "at no time

would involve counter-intuitive absurdities." William Craig, *Theism, Atheism, and Big Bang*

in eternity will we find Tristram Shandy writing, which is absurd, since we supposed him to be writing from eternity. And at no point will he finish the book, which is equally absurd, because for the book to be completed he must at some point have finished." (Craig, *Atheism, Theism, and Big Bang Cosmology*, p. 34). If this turns out to be a valid argument then the correspondence leads to Tristram at any time being finished writing and therefore at no time actually writing and at the same time never finished writing. What makes the situation absurd is not the fact that he will never finish but the claim that at the same time he can be considered finished and therefore not writing for eternity. It is apparent that this paradox arises from the use of the correspondence principle, because it is from this principle that Craig assumes Tristram would be finished writing. The question then becomes whether the principle leads to the conclusion that Tristram at any time could finish writing.

One reason William Craig thinks Tristram should be finished his autobiography by the present is that he has an infinite amount of time to accomplish this task. Craig states that Russell gives this reason; "in order to write about an actually infinite number of days at the rate of one day per year all one needs is an actually infinite number of years." (Craig, *Atheism, Theism, and Big Bang Cosmology*, p. 100). Craig states that because there is a year for every day recorded in the past there ought to be plenty of time available to write about *all* the days.

To complicate the issue William Craig claims that Tristram Shandy will have fallen infinitely far behind. Craig states "At any point in the past or present, therefore, Tristram Shandy has recorded a beginningless, infinite series of consecutive days.

But now the question inevitably arises: *which* days are these? Where in the temporal series of events are these days recorded by Tristram Shandy at any given point?" (Craig, *Atheism, Theism, and Big Bang Cosmology*, p. 101). He concludes that these days must be infinitely distant from the present. If this is true then the interval between the last day recorded and the present is infinite. He states the interval must be infinite because if Tristram has "been writing for an infinite amount of days, that distance will be infinite." (Craig, *Kalam Cosmology*, p. 103). In Russell's version of Tristram Shandy the gap between the last recorded day and his current year of writing progressively gets larger as time passes, although it always remains a finite interval. Craig reasons then that if Tristram has been writing for an infinite amount of time, then that interval between the last day recorded and the present year of writing will be infinite. At no time would the gap between the last day recorded and the present have been finite. For at every moment in the past Tristram has been writing for an infinite amount of time, and therefore Tristram has always been infinitely far behind in his writing.

If Tristram is infinitely far behind this implies that an infinite past entails infinitely distant events. Craig's point is that no matter how far we mentally regress we cannot find the last day written.¹⁸ Craig's thinks we can still establish a correspondence, even though we cannot locate the day that is in correspondence with the present year. Craig notes that the cardinal number of days recorded is $\aleph_0$ and the number of years of writing also has the cardinal number $\aleph_0$. Therefore as they are equivalent sets they can be put in a one-to-one correspondence. This brings up the

medieval problem of traversing the infinite. It appears impossible that Tristram could write an infinite amount of days, and thus traversing an infinite interval, to reach the present. Likewise it seems impossible "for an event which was once present to recede to an infinite temporal distance." (Craig, *Theism, Atheism, and Big Bang Cosmology*, p. 103). The once present event would have had to recede through an endless amount of days to somehow get beyond infinity.

There is no problem with the statement that given an infinite amount of days an equally infinite amount of days can be recorded, especially if it takes Tristram a day to write about each day. However, given an infinite amount of years this does not necessarily entail that the infinite set of days recorded includes all the days in the past up to yesterday (yesterday being the last complete day). Craig states "There is no day of the past to which a past year does not correspond, therefore, assuming that his goal is to record every day of his life, Tristram Shandy should finish the job in an infinite number of years, which, by now, he has had." (Craig, *Kalam Cosmology*, p. 100). It is a valid statement that every past year has a day to which it is in correspondence. However, it is not true that every day has a year to correspond to. For Tristram to be finished he would have to have recorded yesterday, but yesterday requires a future year of writing to be recorded. Therefore yesterday and many other past days do not yet have a year to correspond to, those being future years.

The first issue that we must analyze in depth is whether the principle of correspondence leads to the conclusion that Tristram Shandy would be finished writing at any point. If he could be finished writing then we have the paradox of him

¹⁸ "They all exist in some mysterious way beyond the frontier of past infinity." (Craig, *Theism, Atheism*,

both being finished writing and at the same time so far behind that he will never finish writing. In addition we have the issue of whether there is a finite or infinite interval between the last day recorded and the present. If we decide that Tristram is infinitely far behind we have the additional problem of how he could traverse an infinite interval to reach the present and how a once present event could recede beyond infinity. Phrased in this way we have a clear paradox for Tristram can be described as both being infinitely far behind and finished writing at any point in the past.

(3.2) Quentin Smith Responds

Quentin Smith is open to the possibility of an actual infinite past and enters into a discussion with Craig. Smith takes issue with the claim that the principle of correspondence leads to Tristram Shandy finishing his autobiography. Smith states "it is false that the proposition 'The number of past days written about is the same as the number of past days' entails 'There are no past days unwritten about.'" (Smith; *Theism, Atheism, and Big Bang Cosmology*, p. 87). Smith brings up the notion of a set and its proper sub-set.¹⁹ A proper subset is a subset that does not include all the members of the original set; therefore there is at least one or more member(s) of the original set that do not belong to the proper subset. According to Smith the correspondence principle does not entail that Tristram will ever finish writing. The number of past days recorded is a proper subset of the infinite number of past days, and a "proper subset of an infinite set can be numerically equivalent to the set even though there are members of the set that are not members of the proper subset."

and Big Bang Cosmology, p. 102).

(Smith; Theism, Atheism, and Big Bang Cosmology, p. 87). Just as the "infinite set of natural numbers has the same number of members as its proper subset of even numbers, yet has members that are not members of this proper subset (these members being the odd numbers); so the infinite set of past days Tristram has lived through has the same number of members as its proper subset of days written about, yet has members that are not members of this proper subset (these members being the days unwritten about)." (Smith, Kalam Cosmology, p. 87-88). In Figure 3.1 set B has a last term, that being the last day recorded. From that last term to the present day there will be many days that are not recorded and therefore not part of set B. However, those unrecorded days have been lived through, and are days that Tristram has spent writing, and therefore are in the years that belong to set A. Therefore even though both set A and B are equivalent there are days that are not recorded that do not belong to the set of days recorded. Every time another year of writing is added to set A, a day that was previously unrecorded will be written about and added to set B. Therefore Tristram Shandy will not be finished writing even though both sets are equivalent.

Indeed if we look at Fig. 3.1 we can see that even though there is a one-to-one correspondence between terms in set A and B, set B is instantiated far into the past. In fact if we wanted a more accurate figure set B would be so far into the past that it would not be on the graph at all. Again this is because set A is more temporally extensive than set B, thus with set A a greater duration has elapsed and will reach the present well before set B. Every year of writing has the same temporal extensiveness

¹⁹ "A proper subset is a subset that does not exhaust all the members of the original set, that is to say, at least one member of the original set is not also a member of the subset." (Craig, *Theism, Atheism, and Big Bang Cosmology*, p. 8).

as 365 days recorded. It would take 365 years of writing to record one year of Tristram's life, and by then there would be 364 years worth of unrecorded days. At this rate by the present year of writing, Tristram will be very behind in his progress, and the set of recorded days will be found far into the past. If every year of writing is in a one-to-one correspondence to every day recorded, we can clearly see that the set of years of writing is going to extend far more temporally than the set of days recorded. We can have a correspondence between terms without it entailing that Tristram would be finished writing.

If we do not have to line up the last terms in each set in order to *establish* a correspondence then why would we assume that set B will ever temporally line up with set A? The infinite set of past days has the same number of members as the subset of days recorded but the infinite set of past days has members that are not part of the subset, those being days unrecorded. Therefore although the set of days is equivalent in extensiveness to the set of years spent writing (A), the set of days recorded (B) is not. There are many days in set B (Fig. 3.1) that are not yet part of the set of days recorded that will be paired up with future years. Therefore, even though the sets are equivalent as demonstrated by the correspondence principle, Tristram Shandy will not be finished writing.

Figure (3.1) Craig's Version of the Tristram Shandy Paradox

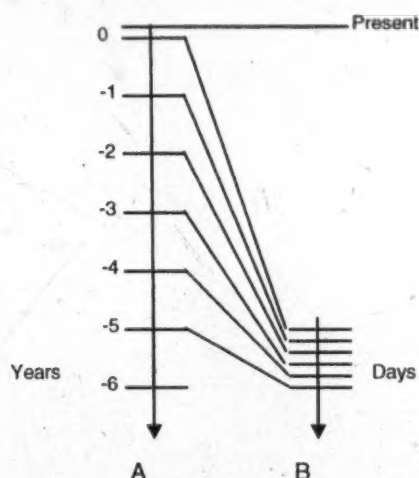


Fig. 3.1. In Craig's version of the Tristram Shandy Paradox there is still a one-to-one correspondence between the terms of set A and the terms of set B. This demonstrates that the set of years spent writing and the set of days recorded are equivalent. This equivalence does not entail Tristram will be finished writing by the present, or finished writing at any point in the past. The days recorded are temporally located far into the past. Every day proceeding the last day recorded will be a day that has yet to be recorded by Tristram Shandy. As a further year of writing is complete another day will be added onto set B. The unrecorded days are days that are not yet part of set B, but are days that Tristram has already lived through. Clearly to be finished writing the last day recorded would have to be yesterday, but yesterday would require at least a year into the future to be recorded. Despite the one-to-one correspondence between terms Tristram Shandy would not be finished writing by the present, but fall progressively farther behind in his work.

(3.3) Ellery Eells

Ellery Eells argues against William Craig by taking issue not with his use of the correspondence principle but with the claim that Tristram would be infinitely far behind. Ellery Eells claims that the interval between the last day recorded and the present year of writing will be finite. Eells notes, there "will always be only a finite number of days between any given day and the day on which he finishes describing the given day." (Eells, p. 454). One reason for this is simply that if we know the last

day recorded and the series ends at the present we have constructed a finite interval with two ends. Since the series is consecutive if the interval has two definite ends the interval will be finite. Eell's states that we could find the day Tristram began writing about his past because the present must be finitely distant from the last day recorded. We are aware that the series of days recorded and the series of years of writing diverge more and more as time passes. Likewise if we regress member by member the gap between the day recorded and the year that Tristram is writing about that day shrinks. As we regress more years we will eventually find the day that Tristram began writing about the day he was then living. However, if we supposed Tristram to be writing for all eternity then there should not be a beginning to his writing. Eells goes to the extreme by stating Tristram could not have always been writing about his past, for before that he must have been writing about his future. If we want to adhere to the premise that he has been writing for all eternity, then we must assume that prior to the first day he began writing about his past he must have been writing about his future. Furthermore, "only after infinitely many years of writing about his future could he have begun to write about his days past." (Eells, p. 454). Therefore if we supposed him to be writing for all eternity, Eells claims, then before that he must have been writing about his future.

It seems that taking a year to write about a future day violates the idea that Tristram is writing about his past and that it takes him a year to record one day. For clearly he would not have been writing about his past for all eternity, he would have been writing about his past for a finite interval and writing about the future for all time before that. We should not be able to find the day Tristram began writing about

the day he is on, and if we did it makes it more absurd to say he is writing about his future. Even if we accept Eells' version it appears we have a further violation of the original argument, for some past days will not be recorded at all. The original argument assumes that Tristram has been writing for all eternity and that it takes him a year to record a day, but since some days are not recorded at all they do not have a corresponding year of writing. Therefore Tristram is not recording an autobiography of his eternal life as the original argument stipulated; instead he records only segments of his life. When Tristram is writing about his future, he is not writing about the existing past days, so clearly it is not true that every day will be recorded.

Furthermore some days may be recorded twice, since some would be when he is recording the future and others when he is recording the past. Eells, in not accepting the infinite series could have infinitely distant events makes the problem more absurd than it otherwise would be. What is important about Eells interpretation is that if Tristram is finitely behind in his work, then we would be able to find the day he began writing about that same day.

(3.4) Robin Small

Robin Small first notes that Russell's version of Tristram Shandy contains no internal absurdities in itself. Russell was attempting to understand what principle applies to infinite collections; the Cantorian correspondence principle or the common axiom that the whole is greater than its parts. Russell did not intend his use of Tristram Shandy to be paradoxical nor did he claim that Tristram Shandy would be finished writing his autobiography. William Craig revises the Tristram Shandy

paradox in order to demonstrate that the correspondence principle does in fact lead to absurdities.

Robin Small takes issue with the claim that a one-to-one correspondence can be established in William Craig's version of Tristram Shandy. With Russell establishing correspondence is not a problem since both sets have determined first members, "the first day of Tristram Shandy's life corresponds to the first year of his writing," (Small, p. 214). Smith states that William Craig uses the principle of correspondence to illustrate that Tristram Shandy would be finished writing and since this is impossible we must reject an infinite past. Small states this argument has its basis in the claim that a one-to-one correspondence can be established. When the paradox is turned into the past there are no first members to the set of years of writing and the sub-set of days recorded. When we try establishing a correspondence from the last members of each set, we cannot locate the last recorded day to match up with the present year of writing. This is blatantly apparent if there are infinitely distant events as William Craig claims, how can you match the last recorded day to the present year of writing if you cannot even locate that day? Small claims "We find then that we cannot locate the day which corresponds to the latest year of writing, of for that matter to any day of writing." (Small, p. 214). Since we find that we cannot locate the last day recorded which corresponds to the present year of writing, we cannot establish a correspondence from the sets last terms. Small concludes it is "*not* the case that 'a one-to-one correspondence between days and years could be established'" (Small, p. 215). Therefore, Small claims, we cannot establish a correspondence and be led to the conclusion that Tristram could be finished his

autobiography. Of course, at the same time we are aware that both sets are denumerable infinities and are therefore equivalent. Small claims that since we cannot establish a physical correspondence, as we cannot locate the last day recorded, then we cannot establish a correspondence at all.

Small claims the failure to establish such a correspondence has more to do with Russell's story not being correctly turned around to suit the past, since "it is still assumed that Tristram Shandy is writing about events which have already taken place." (Small, p. 215). In Russell's version it takes a year for Tristram to write about one past day and the series has a first term but no last term. In Craig's version it also takes Tristram a year to write about a past day but the series has no first term and has a last term. Craig only alters where the beginning and the end of the series is. Small claims this limits our ability to establish a correspondence in Craig's version. So Small gives us a story that would really turn the paradox about. His alternative goes thus: "suppose that Tristram Shandy plans the events of each day in his life in advance, and that it takes him a year to do this for each day. Could he have completed his task up to the present day? If he had been carrying it out through an infinite past, the answer seems to be in the affirmative. For today's events were planned in a certain year, yesterday's events were planned in a previous year, and so on." (Small, p. 215). In Russell's version we could always find a year further in the future to record a further day. Similarly in Small's version we can always find a year further in the past that planned a given day. Again the set of years is equivalent to the set of planned days and are in a one-to-one correspondence. There also is no paradox from the conclusion that Tristram has finished this task up to the present. In addition

we have a set that is infinite but is finite between any two points and therefore there are no infinitely distant events. For the year that planned today was last year. The year that planned yesterday was two years ago. Although the gap between the years of planning and the day it planned increases the farther we recede into the past, the interval remains finite. Similarly in Russell's version the gap between the last recorded day and the year of writing increases as we progress into the future it remains finite as well. Even though the series is finite between points we do not have Eell's dilemma for we never locate the beginning of Tristram's planning. Whatever day we choose there is a year further in the past that planned it. This situation seems to be far better suited for the past, and no paradoxes arise from it. However, "it is true that Tristram Shandy will not be able to continue his planning after a certain day. But it is also true for the original version of Tristram Shandy's work on his autobiography must have had a beginning." (Small, p. 215), and Small's version has an end.

I do not agree with Small's argument that correspondence cannot be established simply because we cannot locate the last day recorded. There is still something distinctly odd with the notion that correspondence entails Tristram will be finished writing. If Tristram was finished writing, then the last day recorded would be yesterday. Yesterday cannot be the last day recorded for that would require a future year. Likewise, if Tristram is infinitely far behind, then we cannot locate the last day recorded to establish a correspondence, then you have to be aware that such a correspondence does not entail Tristram would be finished writing. Clearly if you have to search for the last day recorded then it was not yesterday, therefore logically Tristram cannot be finished writing. However, as the two sets are denumerable

infinities they have equivalent sizes regardless of whether we can physically place them in a correspondence.

There are no infinitely distant events in Russell's or Small's version of the Tristram Shandy illustration, and neither result in paradox. In Craig's version of the Tristram Shandy paradox if we know the last day recorded we can find the day Tristram began to write on. Craig's version when put into the past presupposes a beginning to the writing, as Russell's version had. As Small notes, "The assumption that each day must precede the year of writing that corresponds to it is crucial here: it implies that no correlation between days and year can extend into an infinite past." (Small, p. 216). It does not appear that there would be infinitely distant events in a series that is consecutively ordered and composed successively of discrete members (each member of the series would be understood as having a finite duration). Because the series is composed of finite events with no gaps, then between any two points, there would be a finite interval only. As such Eells would be correct in his analysis that we would be able to find a beginning to Tristram's writing. However, if we use Small's version in which the Tristram Shandy illustration is completely reversed from Russell's then we have a version that does not presuppose a beginning. Therefore Small demonstrates that we can have an infinite series of past events with no infinitely distant events without finding a beginning to the series or resulting in paradox.

(3.5) Summary

William Lane Craig claims that the principle of correspondence leads to Tristram Shandy having finished his autobiography by the present. However, since

the duration of one term of the set of years of writing is 365 times the duration of one term from the set of days recorded, as time progresses the sets diverge more and more and will not be aligned by the present. Secondly, the set of days recorded is the proper sub-set of the set of days Tristram has lived through. As a result there are days that Tristram has lived through that are part of the entire set, but are not included in the proper sub-set. This can be explained in terms of wholes and parts. The whole is equal to the part. For Tristram to be considered finished he would have had to record every part of his autobiography, when in fact we find the proper sub-set does not include every part, as all the unrecorded days are excluded. Thus the sets remain equivalent without the entailment of Tristram finishing his work. Therefore Tristram Shandy will not be finished by the present. In addition to the fact that not every part of the set of days will be recorded we find that its sub-set of days recorded is not as extensive as the set of years spent writing and therefore Tristram is behind in his work.

At this point we also realize that in Craig's version Tristram Shandy cannot be only finitely far behind in his work. If he were only finitely far behind we would be able to locate the last day recorded, and by regressing a year for every day we would be able to determine the day on which Tristram began writing. This directly conflicts with the premise that he has been writing for all eternity. This means that Tristram Shandy must be infinitely far behind. According to Craig, if Tristram has been writing for an infinite amount of time then likewise the interval between the last day recorded and the present will be infinite. This being the case it seems impossible for a once present event to recede day by day to an infinitely distant location from the

present. If it is impossible for Tristram to be infinitely far behind and being finitely far behind determines a starting point then, as Craig concludes, an infinite temporal regress of events is impossible.

Alternatively, Robin Small gives us a version of Tristram Shandy which reverses all aspects of Russell's original version and does not result in any paradoxes. His version demonstrates that an infinite regress of events ordered such that it is finite between points is indeed possible. If Craig's version presupposes a beginning, as Small claims, then we still have the option of having an infinite past that is finite between points without absurdities.

Chapter 4: Problems and Resolutions

The problem of whether Tristram is infinitely or finitely far behind has to do with the ordinal number of the infinite temporal regress of events. Therefore it will be appropriate to discuss Cantor and Russell's ideas on cardinal and ordinal numbers. In addition Quentin Smith and William Craig present a very interesting discussion on whether a series of real objects can be ordered with finitely distant members and re-ordered with infinitely distant members. From this point we can delve into what this entails for the Tristram Shandy paradox.

(4.1) The Basics

If there is a collection of concrete objects, such as balls, we may call that collection a set in which each individual ball is a member of that set. Cantor defines a set as a "collection into a whole of definite, distinct objects of our intuition or of our thought." (Fraenkel, p. 9). That members are distinct refers to the fact that they are different, and therefore cannot be repeated more than once. A member being definite refers to our ability to decide if it belongs to the set, "given a set s , it should be intrinsically settled for any possible object x whether x is a member of s or not." (Fraenkel, p.10).

We can then attach numbers to each member of the set, with a different number for each member, although with each member receiving a number in an arbitrary way.²⁰ Thus we also obtain an abstract set of numbers that is in a one-to-one correspondence with the real objects. Correspondence yields the notion of "cardinal

number' which answers the question 'how many?'; for multitudes of arbitrary objects between which a one-to-one correspondence exists may be defined to possess 'the same cardinal number'" (Fraenkel, p. 24) In addition, as we have previously discussed, we can use correspondence to establish if two separate sets are equivalent or have the same cardinal number. The cardinal number of a set can be defined as "the general concept (universal) which by means of our 'active mental powers' arises from the set S by abstracting from both the special nature of the members of S and the order in which they may occur in S." (Fraenkel, p. 59). The cardinal number then is abstracted from the natural relations between the members and represents how many members are in the set despite their order. By organizing the randomly assigned numbers in order of magnitude we thereby create a succession in the collection of balls. This succession beyond the members, being numbers or balls, is in the order of first, second, third, etc. Because of the one-to-one correspondence the order of the set of balls is the same as the order of numbers. If we disregard the order of the set, the plain collection would determine the cardinal number of the set only, which is simply the number of members in the collection. The cardinal number of balls and numbers would also be the same.

If two sets are equivalent, whether they are infinite sets or finite sets, they have the same cardinal number. There are infinite sets that are not equivalent, as we mentioned in chapter two, and therefore it is possible for infinite sets to have a different cardinal number. The cardinals of infinite sets are called transfinite cardinals. The least transfinite cardinal is called $\aleph_0$ and is the cardinal of denumerable

²⁰ Fraenkel, Abraham A. *Abstract Set Theory* (North-Holland Publishing Company; Amsterdam, 1968).

sets. A set which is "equivalent to the set of all positive integers is called *denumerable* (or countable). If its members are *ordered* according to the magnitude of the integers related to them one speaks of an *enumerated* (ordered) set." (Fraenkel, p. 33). An ordered set that is not enumerated is a sequence that has repeated members. A set is infinite if some subset of it is numerically equivalent to the whole set. And if there are two sets and each is equivalent to the subset of the other, then the sets themselves are equivalent and their cardinals are equal. There are also transfinite ordinals that inform us of the order of succession of enumerated sets. The ordinal of an enumerated set is represented by the symbol ' ω '. The set of whole numbers is an enumerated set that has the ordinal ω : (1, 2, 3, ...). If we add a member to this series we will get the ordinal $\omega + 1$: (1, 2, 3, ... 1) and this sequence is no longer considered enumerated as we have begun to repeat members. As we continue adding we get the ordinal $\omega + \omega$: (1, 2, 3, ... 1, 2, 3, ...).

From the above discussion we can determine that the collection of past days is a collection of distinct and definite members and therefore is a set. Even though the collection of past days is successive and all the days and events do not exist all at once they "are determinate parts of reality, are definite and distinct and can be numbered, they can conceptually be collected into a totality." (Craig, Kalam Cosmology, p. 96). The set of past days is also denumerable and thus has the cardinal $\aleph_0$. The set is also enumerated with the order ω^* : (...-3, -2, -1)

(4.2) The Order of Concrete Objects

Before we engage in a direct discussion of the problems of ordering the sets in the Tristram Shandy paradox it is valuable to discuss the problem in a more general

sense. The main issue of concern here is about a series ordered ω compared to a series ordered $\omega + \omega$. The former implies members that are finitely distant from each other and the latter implies members that are infinitely distant from each other. Both orderings would have the cardinal number $\aleph_0$.

William Lane Craig states we can imagine an actually infinite linearly organized series of balls (Craig, *Time and Infinity*, p. 388). Because these balls are not numbered the ordering of the series is not immediately apparent. We can, as Craig points out, paint on each ball the series (1,2,3,...) or the series (1,3,5,...,2,4,6,...). The first ordering would have the ordinal ω , and the members of such a series would be finitely distant from each other. The second ordering would have the ordinal $\omega + \omega$, and it will have infinitely distant members. These orderings are simply different alternatives for ordering the same series. But regardless of which ordering we choose the cardinal remains $\aleph_0$. Craig suggests that since a series that is ordered ω can be ordered $\omega + \omega$ that the series must have infinitely distant members otherwise the series could not be ordered $\omega + \omega$. Craig seems to be implying that if it is indeed possible to order the series $\omega + \omega$, then there must actually be infinitely distant balls in the original ω series. Therefore he believes that an actual infinite entails infinitely distant events. Therefore $\omega + \omega$ reflects the physical arrangement of balls, not ω . If any collection with the cardinal $\aleph_0$ can have the order $\omega + \omega$, which we can simply stipulate, then it follows that "if an actual infinite exists, it has members infinitely distant from one another." (Craig, *Reply to Smith*, p. 227). And because any series with $\aleph_0$ members can be ordered $\omega + \omega$ they must contain infinitely distant

members.²¹ However just because it is possible to change the order of the series to $\omega + \omega$ this does not seem enough to demonstrate that there are balls that are infinitely distant from one another. Craig states that if we do not like the first ordering "we can wash off the paint and without rearrangement renumber them according to the other type." (Craig, *Time and Infinity*, p. 390). In addition, since the past series of events is $\aleph_0$ then it can be ordered $\omega^* + \omega^*$. To which Craig concluded, "if the past is infinite, there must be events infinitely distant from the present moment. But such events, having once been present, could never temporally recede to their infinite remove from the present event." (Craig, *Time and Infinity*, p. 390). According to William Craig, it is therefore impossible that the past is infinite, for it entails infinitely distant events.

Quentin Smith replies that we can also by the same argument say that a series ordered $\omega + \omega$ *can* be re-ordered ω .²² And since the series *can* be ordered ω then a series ordered $\omega + \omega$ has only finitely distant members. Thus "a series of balls with the order type $\omega + \omega$ has only finitely separated members." (Smith, *Reply to Craig*, p. 112). With the conclusion being that an infinite past is possible. Smith states that Craig's argument rests on the assumption that since an actually infinite series can be reordered $\omega + \omega$, and that is an order which is impossible, then likewise the original ω ordering must also be impossible. From this Craig extends his argument to state that since the infinite past would have infinitely distant events because we can reorder it $\omega + \omega$, then an infinite past is impossible. Smith states, if "the impossibility lies in the

²¹ If the $\omega + \omega$ ordering is possible then there must be infinitely removed ball in the series, "otherwise, the selfsame series could *not* be ordered $\omega + \omega$." (Craig, *Time and Infinity*, p. 390)

²² Smith, "Reply to Craig", p. 112

$\omega + \omega$ ordering rather than the ω ordering, then it is false that a series with an ω can be reordered with $\omega + \omega$ order." (Smith, Reply to Craig, p. 111).

Craig's argument results from the notion that the "distance between members is an objective matter of fact and so does not vary as the ordinal numbers of the members and the order types of the collection vary according to stipulation." (Craig, Reply to Smith, p. 227). And if a series can be ordered $\omega + \omega$ then the objective distance between members must be infinite. Craig states that by reversing the argument as Smith does it winds up with the contradictory conclusion that "if an actually infinite collection exists, it both has and does not have members infinitely distant from one another." (Craig, Reply to Smith, p. 227). That just gives further reason to reject that an actually infinite collection exists.

In the end Craig admits his own argument does not work with the set of concrete objects such as balls.²³ He states there is no guarantee that changing the order of the series will in fact imply that members are infinitely or finitely distant from each other. This is simply because despite the fact that you number three balls (1, 2, 3) there might very well be an infinite distance between the ball numbered one and the ball numbered two. In such a case it would be the case that the number one is infinitely distant from the number three even though the order implies they are finitely distant from each other. Conversely we may have the series ordered (1, 3, 5, ..., 2, 4, 6, ...) which implies there is an infinite distant from five and two. However, the physical distance from the ball numbered five may be only a finite distance from the ball numbered two despite the ordering of the series. This is because "One cannot

assume that the numerical differences between ordinal numbers are isomorphic to the distances between the objects numbered." (Craig, Reply to Smith, p. 228). This is only an issue if there is not a set distances between the objects being numbered. If we are aware that every object has the same distance from each other object and that the series is consecutive then the series would be isomorphic to the ordinal series. Craig states that we could stipulate that the "units of distance be correlated with the ordinal numbers of the balls, ...the third ball is three feet from the beginning, the millionth ball a million feet, the $\omega + 1$ ball $\omega + 1$ feet, and so forth." (Craig, Reply to Smith, p. 229). In such a consecutively ordered series of concrete objects there would be balls that are infinitely distant from the first ball. Craig then notes even if we reorder this series as ω the isomorphism between the distances and the ordinal numbers is broken, because despite the order we imposed there are still balls that are infinitely far apart. (Craig, Reply to Smith, p. 229). If we have an order that is isomorphic to the concrete objects, then if we reorder the series it seems quite likely that that new order will not be isomorphic to the series and therefore does not reflect a change in the concrete objects. Unfortunately we would have to know beforehand which ordering is isomorphic to the series in question.

It seems that the series should be ordered depending on the relation of members in the series. If there are infinitely distant members then the series should be ordered $\omega + \omega$. If the series is finite between members then the series should be ordered ω . Furthermore if the series is ordered ω based on the relations of its members then it cannot be re-ordered $\omega + \omega$ and vice versa. The problem is that it is

²³ Craig, "Reply to Smith", 228

not immediately apparent which ordering an infinite series should have. However, if the series has infinite distant members it becomes apparent that some members cannot be numbered (as there would be no such number), such as the ball immediately after the number one of the second ω ; (...3, 2, 1, ...3, 2, 1). Secondly such an ordering seems odd, since we would never be able to number completely the first ω to start the second. If we are aware of infinitely distant members or infinite intervals within the series then the series is most accurately described by $\omega + \omega$. It is not clear whether a consecutive enumerated series, such as the set of days could have infinitely distant events, without some serious difficulties. In fact because of the above considerations Craig states "we can know a priori that the series of past events, even if infinite, cannot be ordered $\omega^* + \omega^*$ in such a way that the distance intervals are correlated to the ordinal numbers of the events," (Craig, Reply to Smith, p. 229). That is because we cannot guarantee that when we change the order that the "order type correlates with the temporal distances of events from the present." (Craig, Reply to Smith, p. 229). By changing the order from ω^* (...-3, -2, -1) to $\omega^* + \omega^*$ (...-5, -3, -1...-6, -4, -2) does not guarantee that the interval from -1 to -2 is infinite. This argument demonstrates that if infinite time is ordered ω^* then reordering it $\omega^* + \omega^*$ does not guarantee that there will be infinitely distant events. The $\omega^* + \omega^*$ ordering may not be isomorphic to the events themselves. However, when Craig sets up the Tristram Shandy paradox we find that we are led to the $\omega^* + \omega^*$ ordering, because as Eell noticed the ω^* ordering leads to us determining the beginning of the series. If past time can be ordered $\omega^* + \omega^*$ then we face the problem of once present events receding

to an infinite distance from the present. With the Tristram Shandy paradox it seems ω^* and $\omega^* + \omega^*$ orderings are both unsatisfactory.

(4.3) Some Further Concepts from Russell

The series of past days is a denumerable infinite that composes a consecutive series. The fact that the members are consecutive means that any day n is immediately next to day $n + 1$ such that there is no magnitude between. We can have three days, a , b , and c where b is between. There is a "relation of a to b and of b to c , which is not a relation of b to a , of c to b , or of c to a ." (Russell, p. 200). And thus the "Two terms between which the relations in question hold are called *consecutive*." (Russell, p. 201). Day a precedes b , and b precedes c and this ordinal relation reflects the order of the series of days, each day having a predecessor and successor.

The series of past days is also connected. An infinite series that has an end but with no beginning, is not necessarily connected. A collection is connected when "given any two of its terms, there is a certain finite number of steps from one term to the next, by which we can pass from one of our two terms to the other....of any three terms of our collection, one must be between the other two." (Russell, p. 202). This merely illustrates that in the infinite series of days there are no gaps. Thus when a series is not connected "it falls into two or more parts, each being a series for itself." (Russell, p. 239).

The regression of past days can be described as a continuous consecutive series that is connected. This establishes that we have an ordinal relation of terms where a given day immediately precedes another and a given day immediately succeeds another (excluding the end of the series).

(4.4) Infinite and Finite

Is Tristram infinitely far behind or only finitely far behind? An infinite collection does not entail infinitely distant events, but it seems in this case we have no other choice. For if Tristram had fallen only finitely behind we would be able to find the last day he recorded. This year of writing would correspond to that last day recorded. From there we can regress matching every year to every day until we find the day that Tristram began writing on. If we regress 365 days of writing for every day recorded, then the years of writing is regressing larger amounts and eventually will overtake the sub-series until we locate the day at which Tristram began writing. This is illustrated in Fig. 4.1, where we can clearly see that if we can locate the last day recorded we can easily regress a year to a day until we locate the first day recorded. This is impossible since we assumed already that he had been writing for an eternity. Therefore, Tristram must have fallen infinitely far behind by the present moment. This brings up the medieval discussion of traversing the infinite. How can Tristram conceivably write about an actual infinite amount of days to make it to the present day? Perhaps the answer is simply the one Craig mentioned, it would require an infinite amount of years to write about an infinite amount of days. The past is proposed to be infinite, if Tristram never began writing then it is conceivable that he was always infinitely far behind.

Figure (4.1) Tristram Shandy Cannot Be Finitely Far Behind

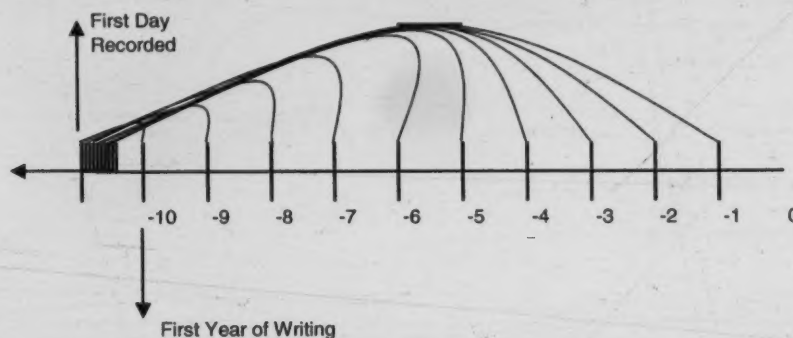


Fig. 4.1. This illustration demonstrates that because every year is 365 times the size of every day if we regress term by term from the set of days recorded and the set of years of writing we will inevitably locate the day on which Tristram began writing. This contradicts our supposition that Tristram has been writing for all eternity, for if he had been writing for all eternity we would not be able to locate the first term of the series. However, if Tristram is infinitely far behind then we would not be able to locate the last day recorded in order to come to this conclusion.

Whether a set has infinitely distant members or finitely distant members depends on how we order the set. Cantor was not only interested in the size of infinite sets "He was also interested in ways of imposing order on to them." (Moore, p. 123). And the cardinal size of an infinite set "has nothing to do with any particular ordering of its members." (Moore, p. 123). Ordinals "are used to measure length, or shape, in the intuitive sense." (Moore, p. 124). Ordinals have one ordinal as the first, and for every ordinal there is an immediate successor, and for each set of ordinals there is an ordinal which succeeds them all. (Moore, p. 125). The ordinal that represents an infinite set is ω .²⁴ But they all have a successor so the successor of ω is $\omega + 1$. And we can proceed until we reach the ordinal $\omega + \omega$, and beyond.

²⁴ "' ω ' is the last letter of the Greek alphabet. It is pronounced 'omega'." (Moore, p. 243)

(4.4.a) Order type ω

Events are not necessarily infinitely distant in an actual infinite collection.

Quentin Smith points this out with the set $(\dots -4, -3, -2, -1, 0)^{25}$. This set has the cardinal number $\aleph_0$ and the order type of regression ω^* . The set is infinite but if we take any number in the set it will be a finite distance from zero. This is simply because if we choose any given member it will only be a finite number of steps before we get back to zero. Therefore we have an infinite set whose members are finitely distant from each other.

It does not seem that the infinite set of days, in the case of Tristram Shandy, is represented by the ω^* order type. If it did then the last day Tristram recorded would be a finite distance from the present year of writing. Then we could locate the day on which he began writing on, which is impossible *ex hypothesi*. It is true that this finite interval would be increasing indefinitely, but it would remain finite. This order does not seem possible for this paradox and yet the order ω^* seems well suited to describe the fact that we have a consecutive series that is continuous, that is connected, and has the ordinal relation we discussed.

(4.4.b) $\omega^* + \omega^*$ Order Type.

If Tristram cannot be finitely far behind then indeed he must have fallen infinitely far behind, which entails the series of past days holds the order type $\omega^* + \omega^*$. This can be represented by the series illustrated in Fig. 4.2. An event occurring at -5 would be infinitely distant from the present. Therefore if -1 was the last day recorded by Tristram then it is infinitely distant from the present year of writing.

Craig states "such events, having once been present, could never temporally recede to their infinite remove from the present event." (Craig, *Time and Infinity*, p. 390). An event at -1 could not recede past the first ω^* (represented by even numbers) to be part of the ω^* (represented by the negative numbers). And an infinite temporal distance would have been traversed.

Figure (4.2) Infinite Regress of Events Ordered $\omega^* + \omega^*$

(...-7, -5, -3, -1...-8, -6, -4, -2 0).

Fig 4.2. illustrates how we can take a series ordered (-4, -3, -2, -1, 0) that is finite between terms and re-order it to be infinitely distant between terms. In this series -8 is infinitely distant from -3. What we end up with here is two separate series both with a first term and lacking a last term. Since the even number series is open ended in the middle of the entire series we end up with this vague area in the middle of the series where we cannot determine the number following -1.

Smith claims that even if we accept that Tristram is infinitely far behind in his writing it does not entail that there is no way to traverse that infinite or that once present events cannot recede to an infinite temporal distance. If we again look at Fig. 4.2, Smith asks how an event located at -5, which was once present, can recede to that infinitely distant point? Smith responds: "by an amount of time with the order type $3 + \omega^*$ elapsing (the times that correspond to -5, -3, and -1 elapse and in addition the times that correspond to the negative even numbers elapse,)" (Smith, *Reply to Craig*, p. 114). He further states "there is no contradiction in the supposition that $3 + \omega^*$ days elapse." (Smith, *Reply to Craig*, p. 114). If an infinite amount of days can elapse in an infinite amount of time with the order type ω^* can elapse then so can $1 + \omega^*$. If a

²⁵ Quentin Smith "Infinity and the Past" (*Philosophy of Science*; 54, 1987)

cardinal infinite with the order type ω^* can elapse then there is no reason to think a larger ordinal number can't elapse with the same cardinal number.

William Craig thinks Smith's response is far too simple. How does event -5 "in receding one day after another, make the transition from the most recent ω^* series to the earlier ω^* series?" (Craig, Reply to Smith, p. 231). If we are talking about a finite collection becoming infinite by successive elapsing of days the response might be in an infinite amount of time and infinite amount of days can elapse, and then of course more can elapse from there.

The next problem Craig perceives with Smith's response needs some consideration. Craig states "when -1 was present, what day was then 'tomorrow'? There was no such day, which is absurd." (Craig, Reply to Smith, p. 231) Does that absurdity arise because an actual infinity is impossible or because the series of temporal events cannot be ordered so? The reason there is no number after -1 in Fig. 4.2 is because a series with the cardinal $\aleph_0$ or the ordinal ω has no immediate predecessor. If it had an immediate predecessor "this would have to be the last of the finite numbers; but we know that there is no last finite number." (Russell, p. 310). Therefore the number succeeding -1 in this series would be the infinityth number and such a number does not exist. Russell notes that if we were to think that ω has an immediate predecessor we end up with a finite collection and not an infinite one. For "suppose n to be the last number before ω ; then n is a finite number, and the number of finite numbers is $n + 1$." (Russell, p. 361), and the series is not infinite after all. To say that there is no immediate predecessor to a series ordered ω is to only notice that there is no last term, it only becomes a problem in a series such as in Fig. 4.2 where

such a series is open-ended in the middle of the regress of temporal events. There is no ordinal number after -1 in Fig. 4.4 because it would be the last term of the first ω^* series. If n is the number after -1 and the last term of the first ω^* series then the series is not ordered $\omega^* + \omega^*$ it would instead be $\omega^* + n$.

Figure (4.3) One-to-One Correspondence in a Series Ordered ω^*

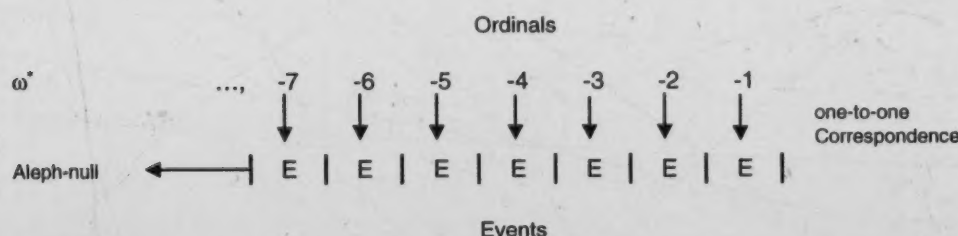


Fig. 4.3 illustrates how we can order the temporal regress of events such that it is finite between terms. The series of events or days un-ordered has the cardinal number $\aleph$. Each ordinal number can be easily placed into a one-to-one correspondence with the events. The ordinal of this series is ω^* . Between any two given points the interval will be finite although the series itself is infinite. For every event there is an ordinal number in correspondence with it.

Figure (4.4) One-to-One Correspondence in a Series Ordered $\omega^* + \omega^*$

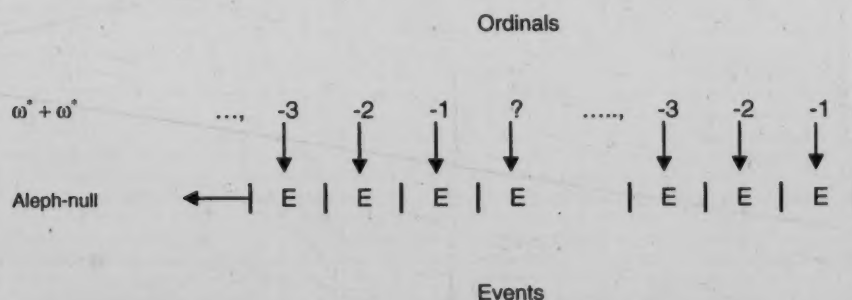


Fig. 4.4. Similar to Fig. 4.3 the cardinal number of events in this series is $\aleph$, however it is ordered with infinitely distant events such that the ordinal number is $\omega^* + \omega^*$. Again the ordinals are in a one-to-one correspondence with the events. However, as we know there are no gaps between the consecutive series of events we know there is a day immediately following the event with the ordinal -1 but we cannot determine the ordinal number that is in correspondence with that event. Having an ordinal series that is open ended in the middle of the entire series of temporal events makes the series not connected where we can no longer determine the ordinals for some events. We do know that for every event in the series there is a day immediately preceding and immediately following it, with the exception of the first term. Therefore, despite the fact that we cannot know the ordinal for the event following -1 of the second series, we do know that there is a day there. This implies that although the series is ordered as to have infinitely distant events the series of events actually is finite between points.

From Fig. 4.2 we can see that one infinite series is closed at the present and open infinitely into the past. We then have a second infinite series with a first member and also open into the past. What complicates the situation is having a series open ended in the middle of the whole series of temporal events. By ordering the series this way it seems the series is no longer connected, for we have two separate series and cannot tell what the integer after -1 would be. However, just because we have no ordinal to assign the day after -1 does not mean there is no day after the day with the ordinal -1. In the consecutive series of days there should be a day immediately after the day with the ordinal -1. Each day is of a finite duration and located a finite distance from the day immediately before and after it. Therefore if the series is to remain consecutive there will be a day immediately following the day assigned the ordinal -1, that following day, however, does not have an ordinal number assigned to it. The fact that there is no ordinal for that day, might mean that the $\omega^* + \omega^*$ ordering is not isomorphic to the series of days, not that there is no day following -1. From our discussion at the beginning of the chapter we noted that we first have a set of concrete members, and then we create a set of abstract integers placed in a one-to-one correspondence to illustrate the ordinal number. In Fig. 4.3 we can see clearly that there is a one-to-one correspondence between every day and every integer. The series in 4.3 is ordered ω^* and is a connected and consecutive series. In Fig. 4.4 we can see Craig's problem, for what is the day after -1? The problem is not that that day does not exist or even that we cannot find it. The problem is that the day after the day with the ordinal -1, does not have a clear ordinal number. From the end of the first ω^*

to the beginning of the second ω^* is a vague space where there is no last term to the first ω^* ordering and therefore no definite number.

We can call the day after -1 day x as it has an unknown ordinal number, and the day after that $x + 1$. If we proceed from x to the present we have constructed a series with two ends and therefore have constructed a finite, not an infinite interval. Therefore the order $\omega^* + \omega^*$ may imply infinitely distant events, but the series of days does not have infinitely distant events, therefore the series of integers does not match the series of days.

It is interesting to note that the $\omega^* + \omega^*$ is arbitrarily correlated with the series of concrete objects. A series that can be ordered $\omega^* + \omega^*$ can also be ordered $\omega^* + \omega^* + \omega^* + \omega^*$ without changing the cardinal number. This would mean that not only can we not locate the last day recorded but we do not know which ordering it is under. With the series ordered $\omega^* + \omega^*$ we seem to arbitrarily stipulate that -1 in Fig. 4.2 is the last recorded day and the first member of the second infinite ordering, thus making -1 infinitely distant from the present. The last day recorded might be the day before -1, or the day before that, in which case the last day recorded would be -2, or -3 making the ordering $\omega^* + 2$ or $\omega^* + 3$. In fact we cannot determine the day that should be labeled -1, the first day that somehow becomes infinitely distant from the present. Nor can we determine which day is the last day recorded, of those that are infinitely distant from the present. Because any such day would be beyond infinity and we would be unable to determine when the second series begins, assuming the second series exists. We cannot physically determine or locate the first day that is at an infinitely distant duration from the present, nor can we therefore determine that the

first day that is infinitely distant from the present would be the last day recorded by Tristram Shandy.

There is no infinity number following -1 , indeed we cannot decipher what that next member would be. I think this is an important problem with there being an infinite distance between Tristram's last day recorded and his present year of writing. If -1 is the ordinal number representing that last recorded day, it makes no sense that we could not decipher what the following day is. Especially since the following day would have the relation of immediately following the previous day, and is consecutive. The following day is obviously the first unrecorded day. The last recorded day and the first unrecorded day are located a finite distance away from each other, in that one immediately precedes the other. If we know that -1 is the last recorded day, then we know that the very next day is the first unrecorded day. If we know the first unrecorded day then it would be a finite interval from the present. The problem is if we know that -1 represents the last recorded day, then how can we not know the very next day? How can there be no next day? The fact is that in a consecutive series of days there is a day immediately succeeding the day with the ordinal -1 , however, as that day would have the ordinal reflecting the last term of ω^* there is no ordinal to assign to it.

Craig states that if Tristram is infinitely far behind then it is impossible that a once present event could recede to an infinitely distant event or impossible that the infinite can be traversed. Despite Smith's response it does in fact seem impossible that an event that was once present could recede an infinite distance away from the present. However, this impossibility seems to arise from the fact that Tristram can

only be finitely far behind, not that he is infinitely far behind and would not have been able to traverse the infinite. We have already demonstrated that by ordering the series $\omega + \omega$ we lose members of a consecutive series. It also seems impossible that Tristram could have fallen infinitely far behind in the first place. This can be demonstrated from Craig's own argument against an actual infinite being created by successive addition.

Craig argues that it is impossible to form an actual infinite by successive addition because he wants to argue that even if an actual infinite is possible time could not be so considered since it is formed by successive addition. Craig states that the series of past events would be an actual infinite because "at any point in the past the series of prior events remains infinite and actual." (Craig, *Kalam Cosmology*, p. 97). The reason he gives for the impossibility of an actual infinite created by successive addition is that "for every element one adds, we can always add one more. Therefore, one can never arrive at infinity. What one constructs is a potential infinite only, an indefinite collection that grows and grows as each new element is added." (Craig, *ibid*, p. 104). And a potential infinite cannot be turned into an actual infinite by simply adding more members. Craig also adds the "only way a collection to which members are being successively added could be actually infinite would be for it to have an infinite 'core' to which additions are being made." (Craig, *ibid*, p. 105). What Craig appears to be forgetting in his argument against an actually infinite past is that the past is a series without a first member. At no time in the past was the past a finite collection and therefore at no time in the past could the series of time be

considered potentially infinite. This does not mean there was an infinite 'core', only that no matter how far we regress into the past the past remains actually infinite.

However, it is from the above argument that we can see how it is impossible that Tristram could have fallen infinitely far behind. The problem is how an event that was once present could recede to an infinitely distance from the present. If that event, whether it be the last day recorded or the first day unrecorded, was once present then Tristram would be constructing a potential infinite from that point into the future. From any present event into the future would be a potential infinite because the present event would be a first member increasing indefinitely into the future but always being finite at any point. Therefore if an actual infinite cannot be formed by successive addition it follows that it is impossible that the present could be infinitely distant from any event in the past that was once present. No matter how many years Tristram writes he does not write more than a finite amount from the event that was once present, for the potential infinite is only a finite collection that increases indefinitely. This argument has nothing to do with the various problems with ordering a series $\omega^* + \omega^*$ or the fact that we would not be able to find the last recorded day in order to establish such an order. This argument instead is that despite the ordering of the series we can only create a potential infinite from the once present event progressing into the future. We may order the series as a regress (...-3, -2, -1) but despite that order we are aware of the fact that time is successive and progresses towards the future, and a series starting from a once present event could not progress from a potential infinite into an actual infinite. As Craig states " $\aleph_0$ has no immediate predecessor. Therefore, one can never reach $\aleph_0$ by successive addition or counting,

since this would involve passing through an immediate predecessor to x_0 ." (Craig, *Atheism, Theism, and Big Bang Cosmology*, p. 31).

William Craig would agree that it is impossible for time to have infinitely distant events for the reasons I discussed. From this, however, he would conclude that an actually infinite past is impossible, not that the series ordered with infinitely distant events is impossible.

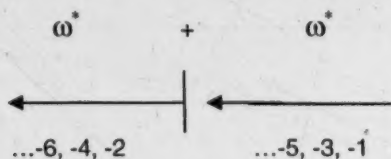
(4.5) Possible Alternative Orderings

We previously rejected the $\omega^* + \omega^*$ ordering because we lost members of a consecutive series. By rejecting this ordering we came to the conclusion that Tristram Shandy is not infinitely far behind. We supplemented this discussion with an argument to demonstrate that Tristram could not have fallen infinitely far behind in the first place. By accepting these arguments we no longer had to deal with the problem of an event receding to an infinitely distant location from the present. We can imagine $\omega + \omega$ orderings, however, that do not violate the principle that time is a consecutive series but still have the problem of infinitely distant events. To make this analysis completely thorough we will have to examine these possibilities. In addition, we should closely examine the validity of Small's version of Tristram Shandy to see if his ω ordering results in any contradictions.

(4.5.a) Time Before Time

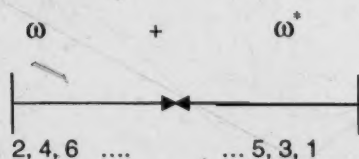
Originally we described the past with infinitely distant events as having the ordered represented in Fig. 4.5. This series has a last term (the present) and regresses infinitely into the past, then an event that is infinitely distant from the present begins the second ω ordering and this regresses infinitely into the past.

Figure (4.5) Time Ordered in Craig's Tristram Shandy Paradox



Thus each ω^* section of the series has a last term with no first term. As we previously discussed this series breaks our principle that time is consecutive. What happens if we alter the end of the second ω^* regress so that the series looks like this?

Figure (4.6) an Alternative Ordering For Time



This series is still consecutive, so we find we cannot reject the $\omega + \omega^*$ ordering based on that issue. We still have the problem of infinitely distant events. In this version instead of losing members we have an incredibly dense center, such that neither series ever connects, but every day has an immediate predecessor and successor. This ordering may be an improbable suggestion to how time is ordered, but it is a possible ordering none the less²⁶. We could now claim that time had a beginning and that it has infinitely distant events as well. This section is called time before time, because as the past is infinite, we would never be able to find the second section of time (ω)

²⁶ Originally I had rejected this possibility because I thought that such an order was impossible. Bernard Linsky pointed out that although an unlikely option it was still a possibility. In particular, he pointed out that this ordering is still consecutive and therefore cannot be disregarded on that basis.

by regressing from the present, likewise from the beginning of time we would never be able to progress into the first section of time (ω^*). Therefore, no matter how far we progress in ω , and no matter how much we regress is ω^* , we never find where they meet, since they never do. The question arises whether time can be ordered so, with infinitely distant events and without contradiction.

William Lane Craig would reject this ordering of time. Even though it suggests that time has a beginning it also suggests that time is infinite. Since William Craig thinks that infinite time suggests infinitely distant events, and that a series with infinitely distant events is impossible, he would reject this ordering. I agree that infinitely distant events are impossible which is why time cannot be ordered so. In addition I want to demonstrate that time without a beginning is conceivable, not that time with infinitely distant events and a beginning is conceivable. Therefore I will have to demonstrate that this series that has both infinitely distant events and a beginning is problematic.

This series brings up the problem of traversing the infinite again. According to this new ordering, if Tristram began to write in ω he would never make it from ω to ω^* . Since the ω series has a beginning (a beginning to time and Tristram's writing) no matter how much Tristram wrote he would only be traversing a finite interval. He would never be able to traverse the infinite, or he would never be able to complete the first infinite to reach the second. This is because the ω series has a beginning and therefore Tristram would only be creating a potential infinite, he would be able to write indefinitely, never completing more than a finite interval. This is similar to Russell's version of Tristram Shandy in which there is a beginning to Tristram's

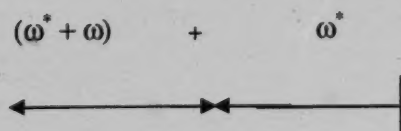
writing and the series progresses into the future. Again we have the problem of creating an actual infinite by successive addition, what is indefinitely increasing never becomes an actual infinite by successive addition. Some people may not be convinced by this response. They may reply that Tristram could write about an infinite amount of days if he had an infinite amount of time. By writing one day beyond that he would have traversed ω into ω^* . However, would he reach ω^* ; given that that series has no last member, and therefore no last day for him to reach in order to traverse the one series into the other? Let us suppose for arguments sake that Tristram could write an actually infinite number of days, even though he had a beginning to his writing and this would require moving from a finite amount to an actual infinite by means of successive addition.

By the present Tristram could not have only recorded a finite interval of ω . Having lived through $\aleph_0$ years he has by now recorded an equal $\aleph_0$ number of days. Any point in ω to the beginning of the series is a finite interval. Thus the last day recorded cannot be found in ω . The last day recorded must be found in ω^* . Any point in ω^* is a finite distance from the present. Thus the last day recorded is finitely distant from the present. Despite the fact that there are infinitely distant events, Tristram will only be finitely far behind in his autobiography. Our argument that we can locate the first day recorded when Tristram is finitely far behind applies here. If we find that the last day recorded is a finite distance from the present we can determine the day Tristram began writing. Since every day in ω^* is finitely distant from the present, the first day recorded will be finitely distant from the present and found in ω^* , not ω . Therefore the first day recorded and the last day recorded both are found in ω^* . This

contradicts the fact that we stated that Tristram began with the first term of segment ω . Furthermore it suggests that since the beginning of Tristram's writing is found in ω^* , Tristram has not recorded any day in ω^{27} . Again having infinitely distant events creates contradictions. In this case even though we stipulate the order of the series we find the beginning to Tristram's writing is not where we anticipated, and thus between any year of his writing and any day he recorded is a finite distance. Even though we order it with infinitely distant events, we find that is not the case, in fact we find ω disappears all together. Again we are led to the conclusion that time does not have infinitely distant events and is connected despite how we order the series. To exclude orderings such as this we can expand our principle that the series of time is consecutive. In order for a series to be considered consecutive every member must have a member immediately preceding and succeeding it. In addition to this we can now add that every segment of time has an immediate predecessor and successor. Clearly ω has no immediate successor and ω^* has no immediate predecessor.

Figure (4.7) A Second Alternative

Similarly to the order above we can order the series thus:



The principle we included to exclude the ordering in 4.5.b similarly excludes this ordering. However, if we do not want to exclude either based on a principle then this

²⁷ Martin Tweedale pointed out the contradictions inherent in this ordering during our discussion of possible alternative $\omega + \omega$ orderings.

can be a valid option. This $\omega^* + \omega + \omega^*$ ordering has no beginning and infinitely distant events. If we want to accept this ordering then we have to accept the traversal problem associated with infinitely distant events in addition to time having incredibly dense sections. This ordering is infinite and has no beginning and therefore for the purposes of this paper does not exclude an infinite past. If however we want to illustrate that it is precisely the $\omega^* + \omega^*$ ordering that causes paradoxes to occur with an infinite past then this ordering should also cause problems.

Figure (4.8) Correspondence Illustrated by Second Alternative Ordering

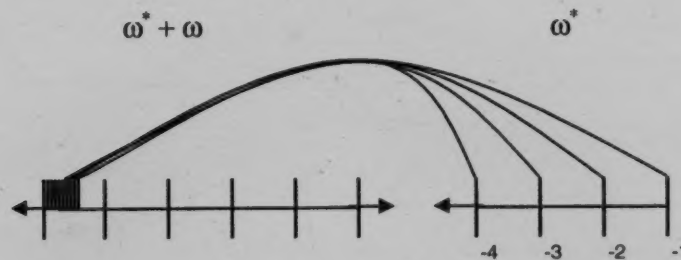


Fig. 4.8 illustrates the correspondence between years of writing and days recorded in this alternative $\omega + \omega$ ordering.

Initially if we look at Fig. 4.8 it does not appear that we have a problem with this ordering when applied to the Tristram Shandy illustration. Every year corresponds to every day, and the series is still consecutive. It also seems that every year of writing is infinitely distant from every recorded day. If we accept that this is satisfactory we still accept that time can be infinite and we also accept that the $\omega^* + \omega$ ordering is not necessarily impossible. That is, it would not be impossible to order time this way, but we still have the traversal problem to deal with. Tristram will never record all the days up to the present, for he will never traverse the infinite to record any day in ω^* (the section that ends at the present). This is precisely the

traversal problem that is created with the $\omega^* + \omega^*$ ordering that makes William Craig think that time cannot be infinite (as his paradox rejects the ω^* ordering as well).

What stands out in this ordering is that neither series $[(\omega^* + \omega) + \omega^*]$ ever meets the other. Even though we can correlate every year of writing with every day recorded, the years of writing are found in ω^* and the days recorded are found in $\omega^* + \omega$. We never find years of writing in $\omega^* + \omega$ that correlates with days recorded in $\omega^* + \omega$. If we did find a year of writing in $\omega^* + \omega$ that can be paired up to a day recorded in $\omega^* + \omega$ then from there we would be able to determine the beginning of his writing, where there should be none (re-enforcing our concern that the Tristram Shandy illustration presupposes a beginning). This is because $\omega^* + \omega$ has no infinitely distant events, as soon as we can determine a year of writing in $\omega^* + \omega$ that corresponds to a day recorded in $\omega^* + \omega$ then we can consequently determine a beginning. This is clearly absurd, if he has lived through all of ω^* and $\omega^* + \omega$, and he has been writing for all that time, then $\omega^* + \omega$ should contain an infinite amount of years he has been writing. But we never find those years because ω^* is infinite into the past, and therefore there are in ω^* an infinite amount of years to correspond to an infinite amount of past days. Again we find that the $\omega + \omega$ ordering makes things absurd especially when combined with the Tristram Shandy story.

It is interesting to note that if we apply either of our alternative $\omega + \omega$ orderings to Small's version, which does not presuppose a beginning, no problems arise. His series is ordered ω^* , and therefore his version takes place entirely in ω^* . There is always a further year in the past of ω^* that planned a day in ω^* . We never reach ω^*_2 or even determine that it is there. Every $\omega^* + \omega^*$ ordering causes

absurdities. When the $\omega^* + \omega^*$ ordering is applied to Craig's Tristram Shandy illustration often we can determine a beginning to Tristram's writing or are led to further absurdities. Small's version is ordered ω^* and has none of the problems associated with the $\omega^* + \omega^*$ orderings, and his illustration does not presuppose a beginning.

Therefore there is no $\omega^* + \omega^*$ ordering of Tristram Shandy that does not lead to absurdities. Furthermore if the Tristram Shandy illustration is finite between points it invariably leads to a beginning. Even when the series is ordered $\omega^* + \omega^*$, if we find both the years of writing and the days recorded in one ω^* section we can from there determine a beginning. Therefore the Tristram Shandy illustration either has a beginning when ordered ω^* , or leads to absurdities when ordered $\omega^* + \omega^*$. By comparing Small's ω^* ordering to Craig's ω^* ordering we will be able to see why Craig's leads to a beginning and Small's works.

(4.5.c) Small's Version Compared to Craig's

In Robin Small's version of Tristram Shandy Small completely reverses Russell's original illustration. Small's version demonstrates that time can be ordered ω^* and have no beginning without any contradictions. However, if it turns out that his version does entail paradoxes we must conclude that time cannot be infinite. We have demonstrated that the past cannot be ordered $\omega^* + \omega^*$. We have also shown that Craig's Tristram Shandy Paradox presupposes a beginning, which is why if time is ordered ω^* we find a beginning where there should not be one. Therefore, if Small's version turns out to be contradictory we will have demonstrated that infinite time ordered ω^* or $\omega^* + \omega^*$ is impossible. If we demonstrate that Small's version is

acceptable then we can conclude that time ordered ω^* is possible and time can be infinite.

Small asks us to imagine that Tristram has planned every day of his life in advance. It takes him a year to plan every day. Clearly Tristram would be finished by the present, and there is no paradox from this fact. Every day recorded corresponds to a year of planning, and the sets are equivalent. The series is consecutive, finite between points and infinite into the past. With William Craig's version of the paradox, by having the members finitely distant from each other enabled us to determine a beginning. In figure 4.9 we can see this result does not occur in Small's version. This is because the days in Fig. 4.9 and the years both begin at the present and regress into the past. We can see, by looking at Fig. 4.9, that the sets diverge more and more, and we will never determine a beginning to this series. The reason Small's version works is precisely the reason why in Craig's version Tristram Shandy cannot be finitely far behind in his writing. For this is the exact reverse. Whereas in Craig's version we find a beginning by regressing members, in Small's version we merely find the end by progressing forward.

Figure (4.9) Small's Version of Tristram Shandy

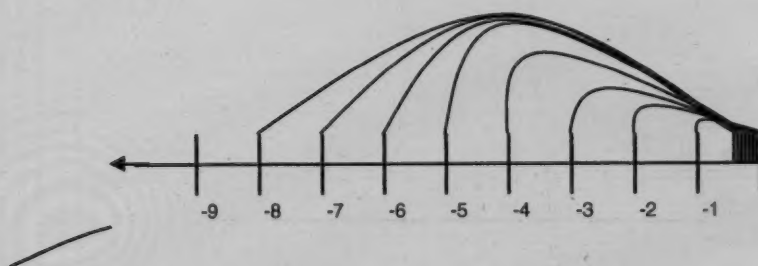


Figure 4.9 illustrates Robin Small's version of Tristram Shandy in which Tristram takes a year to plan the events of a

single day. In this illustration we can see that every year correspond to every day planned, and he is finished planning all the days up to the present. As we regress down the series the interval between the year of planning and the day it plans grows, but remains at any time a finite interval.

In Craig's version the fact that the set of years of writing was more temporally extensive than the set of days recorded caused a problem. The gap between the last day recorded and the year that recorded it diverged more and more. If this gap was a finite interval we found a beginning to Tristram's writing. If the gap was an infinite interval we had the problem of infinitely distant events.

In Small's version even though the years of the planning set diverge more and more from the days planned it never becomes an infinite interval. As we regress member by member the interval between a year of planning and the day it plans only indefinitely increases. In addition every day is planned, for there is always a year into the past that planned it. This is similar to the fact that for every day recorded there is a year of writing in the future in Russell's version. As the interval between days that were planned and the year that planned those increases as we regress, we will never find a beginning to this series. There will always be another day, and a year to plan that day further into the past. And as we *progress* along the series the interval between a year of planning and that day it plans shrinks, until we arrive at the present, where the last day planned and the year the planned it both end. From the fact that in Small's version we do not find infinitely distant events nor do we find a beginning to the series we can conclude the paradoxes found in Craig's version lie in the basic structure of the Tristram Shandy Paradox and not from the impossibility of an infinite past.

Craig does not change any element in Russell's version of the paradox, except the fact that the series has a beginning. By stating that Tristram has been writing for all eternity, but not altering the original illustration to describe this, he creates the paradox he wants to demonstrate. We find a beginning because Russell's version had a beginning, and this was not altered.

Figure (4.10) Russell's Version of Tristram Shandy

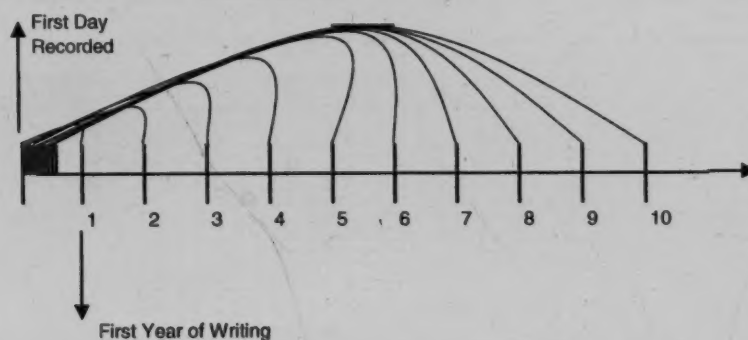


Figure 4.10 illustrates Russell's version of the Tristram Shandy Paradox. We use this illustration to demonstrate how William Craig fails to alter important aspects of this illustration when he turns it into the past. To alter this figure to represent Craig's version of the paradox we need only to make year ten the end of the series, and open the series into the past.

Figure 4.10 illustrates Russell's version of the Tristram Shandy illustration. As we can see this figure greatly resembles Fig. 4.1 which we used to demonstrate that Tristram cannot be finitely far behind. As we can see all that we need to alter in order for this Figure to represent Craig's version is to change the ends of the series. Instead of having a first term and no last term, we need only to make it so that there is a last term with no first term. Using Fig. 4.10 we simply have to stipulate that year 10 is the present, and make it open ended into the past, and then we will have a Fig identical to Fig. 4.1. In addition Craig simply stipulates that Tristram has been

writing for all eternity. Because Craig altered no other aspects of this illustration we can still find a beginning to his writing. Therefore it is Craig's illustration itself that presupposes a beginning, and the premise that he has been writing for all eternity which makes this contradictory. From this we are forced to disregard the ω^* ordering for the more absurd $\omega^* + \omega^*$ ordering and we find that no $\omega^* + \omega^*$ ordering is without problems. Figure 4.10 also illustrates what happens when we progress into the future in Russell's version; the sets diverge more and more. Naturally, if we alter it to represent Craig's version the sets become closer together the more we regress.

Small uses his version to demonstrate how we can easily establish a correspondence. For today is the last day planned and it corresponds to last year, the year that planned today. I use this to demonstrate that we can have a series ordered ω that is consecutive and infinite into the past. By comparing Craig's version to Small's version we realize that the reason Craig's version leads to the contradiction of finding a beginning when there should be none arises from the illustration itself and not in the impossibility of an infinite past ordered ω^* .

From this we can conclude that it is consistent to assert that past time is a consecutive and connected series. For every day and for every interval there is a day preceding and succeeding it. Therefore the past does not have infinitely distant events. Furthermore the past can still be ordered ω^* and thus time can be finite between points and have no beginning without contradiction.

(4.6) Summary

In this section we have shown that no ordering of time that has infinitely distant events is without significant problems. In Craig's version of Tristram Shandy

if the series is ordered $\omega^* + \omega^*$ it is no longer a consecutive series. We also demonstrated that it is impossible that Tristram could have fallen infinitely far behind in the first place. Furthermore, we explored two alternative orderings with infinitely distant event that are also unacceptable. We must, therefore, agree with William Craig that an actual infinite past time with infinitely distant events is absurd. If it is impossible for infinite past time to be ordered with infinitely distant events then clearly infinite past time does not have that order. This in addition to the fact that the past cannot be ordered ω^* under Craig's version of Tristram Shandy leads Craig to the conclusion that an actual infinite temporal regress is impossible. However, the problem lies with Craig's version of Tristram Shandy and not in the ω^* ordering. Small clearly gives us an example of Tristram Shandy where the past can be ordered ω^* without problems arising. Craig's Tristram Shandy illustration does succeed in proving that time ordered with infinitely distant events is impossible, but does not succeed to proving that an infinite temporal regress is impossible. Therefore an actually infinite temporal regress of events can be ordered ω^* without contradiction.

Chapter Five: Conclusion

Russell initially designed his Tristram Shandy example to illustrate that problems arise from the axiom the whole is greater than its parts and not from the principle of correspondence. With Russell's example we demonstrated that Tristram Shandy would not be finished writing and that it is still consistent to state that no day will remain unwritten. Therefore, Russell's use of the correspondence principle leads to no paradoxes. William Lane Craig revises Russell's account of Tristram Shandy, and places it into the past in order to bring out the paradoxes he claims arise from the use of the correspondence principle.

We have shown that the correspondence principle does not entail that Tristram Shandy will finish writing by the present. As Smith noted the set of days recorded is the proper sub-set of the set of days lived through and thus can be numerically equivalent to the set while the whole set has members that the sub-set does not. There are days that Tristram has lived through, we found, that are not yet recorded that are not part of the proper sub-set. Secondly, if we pay attention to the extensiveness of the sets we are then aware that the set of years of writing encompasses a greater duration than the set of days recorded, and therefore the set of days recorded is located far into the past.

We then determined that Tristram Shandy cannot be infinitely far behind in his autobiography. We found every ordering that involves infinitely distant events is

paradoxical. We also noted that Craig's version of the paradox the past cannot be ordered ω^* . By analyzing the similarities found in Russell's version compared to Craig's version of Tristram Shandy we realize that Craig's illustration presupposes a beginning. William Craig fails to reverse all aspects of Russell's version and thereby creates an example that inevitably leads to a beginning just as Russell's example originally has a beginning. Small reverses Russell's Tristram Shandy example completely and thereby proves that ordering past time ω^* does not result in problems.

Therefore I would conclude that the Tristram Shandy paradox does not demonstrate that an actually infinite past is impossible. Firstly it does not demonstrate that the Cantorian principle of correspondence leads to absurdities when applied in reality; in particular it does not lead to Tristram being both finished writing and being infinitely far behind. Even when it comes to the problem of traversing the infinite I think we can demonstrate that ω^* describes the series of days, and that the ordering $\omega^* + \omega^*$ is in fact impossible without re-arrangement of the concrete members of the series. From this I conclude that the paradox presupposes a beginning, and that we can have an actually infinite past that is ordered ω^* without contradiction. Therefore if time is infinite it is a consecutive series ordered ω^* . The Tristram Shandy Paradox does not demonstrate that an actual infinite past time is impossible, only that time ordered $\omega^* + \omega^*$ is.

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